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Colenso, John William
The elements of algebra

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# A
# KEY
# TO THE
# LEMENTS OF ALGEBRA.

## PART II.

BY

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# A
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# ALGEBRA:
## PART II.

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THE REV. J. W. COLENSO, D.D.
BISHOP OF NATAL.

NEW EDITION.

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# KEY TO PART II.

⁂ There are a few solutions, (to which a star is prefixed,) which correspond to questions given in the *Octavo* Edition, but not in the Duodecimo. This has arisen, in some cases, from the same question having been accidentally repeated in the latter—in others, from the question being considered for some reason defective, and replaced by a better one in the larger Edition, to which the Key exactly answers.

*The bracket* ( ) *will be used to refer to Articles in Part I,*
[ ] *to Articles in Part II.*

## Ex. 4.

7. Let $\frac{a}{b} = x = \frac{c}{d}$; then $\frac{a}{b} = x = \frac{(b+d)\,x}{b+d} = \frac{a+c}{b+d}$,

$$\text{or } = \frac{(b+md)\,x}{b+md} = \frac{a+mc}{b+md}:$$

let $\frac{a}{c} = y = \frac{b}{d}$; then $\frac{a}{c} = y = \frac{(c-d)\,y}{c-d} = \frac{a-b}{c-d}$, &c.

8. $\frac{c^2}{d^2} = x^2 = \frac{(b^2+d^2)\,x^2}{b^2+d^2} = \frac{a^2+c^2}{b^2+d^2}$, &c. : $\frac{b^2}{d^2} = y^2 =$ &c.

## Ex. 9.

1. Observe to assume $\sqrt[3]{(22 \pm 10\sqrt{7})} = x \pm \sqrt{y}$, not $\sqrt[3]{(10\sqrt{7} \pm 22)}$.

7. Let $\sqrt[3]{\{m\,(\frac{7}{4} - \frac{5}{4}\sqrt{2})\}} = x \pm \sqrt{y}$;

$\therefore x^2 - y = \sqrt[3]{\{m^2\,(\frac{49}{16} - \frac{25}{8})\}} = \sqrt[3]{(-\frac{1}{16}m^2)} = -\frac{1}{4}$, if $m = \frac{1}{2}$;

$\therefore y = x^2 + \frac{1}{4}$; but $\frac{7}{4}m = x^3 + 3xy$, or $4x^3 + \frac{3}{4}x = \frac{7}{8}$;

$\therefore x = \frac{1}{2}$, $y = \frac{1}{2}$, and $\sqrt[3]{\{\frac{1}{2}\,(\frac{7}{4} \pm \frac{5}{4}\sqrt{2})\}} = \frac{1}{2} \pm \sqrt{\frac{1}{2}}$,

or $\sqrt[3]{(\frac{7}{4} \pm \frac{5}{4}\sqrt{2})} = \frac{1 \pm \sqrt{2}}{2\sqrt[3]{\frac{1}{2}}} = \frac{1 \pm \sqrt{2}}{\sqrt[3]{4}} = \sqrt[3]{\frac{1}{4}} \pm \sqrt[6]{\frac{1}{2}}$.

8. Let $\sqrt[3]{\{m\,(22 \pm 9\sqrt{6})\}} = x \pm \sqrt{y}$;

$\therefore x^2 - y = \sqrt[3]{(-2m^2)} = -2$, if $m = 2$;

$\therefore y = x^2 + 2$; but $22m = x^3 + 3xy$, or $2x^3 + 3x = 22$;

$\therefore x = 2$, $y = 6$, and $\sqrt[3]{\{2\,(22 \pm 9\sqrt{6})\}} = 2 \pm \sqrt{6}$,

$$\text{or } \sqrt[3]{(22 \pm 9\sqrt{6})} = \sqrt[3]{4} \pm \sqrt[6]{54}.$$

9. Let $\sqrt[3]{\{m\,(25 \pm 22\sqrt{2})\}} = \sqrt{x} \pm \sqrt{y}$;
$\therefore\ x - y = \sqrt[3]{(-243m^2)} = -7$, if $m = 1$;
and $2(x + y) = \sqrt[3]{(1593 + 1100\sqrt{2})} + \sqrt[3]{(1593 - 1100\sqrt{2})}$
$= \sqrt[3]{3148.62} + \sqrt[3]{37.38} = (14 + f) + (4 - f) = 18$, &c.

10. As before, we get $x - y = \sqrt[3]{(-54m^2)} = -6$, if $m = 2$;
and $2(x+y) = \sqrt[3]{\{4\,(296 + 110\sqrt{7})\}} +$ &c. $= \sqrt[3]{2348.108} + \sqrt[3]{19.892}$
$= (13 + f) + (3 - f) = 16$; whence $x = 1$, $\sqrt{y} = \sqrt{7}$,
and $\sqrt[3]{(11 \pm 5\sqrt{7})} = (1 \pm \sqrt{7}) \div \sqrt[3]{m} = (1 \pm \sqrt{7}) \div \sqrt[3]{2}$.

11. As before, we have $x - y = \sqrt[5]{(32m^2)} = 2$, if $m = 1$;
and $2(x+y) = \sqrt[5]{215231.95} + \sqrt[5]{.047} = (11 + f) + (1 - f) = 12$; &c.

12. As before, we have $x - y = \sqrt[5]{(-32m^2)} = -2$, if $m = 1$;
and $2(x+y) = \sqrt[5]{23167.616} + \sqrt[5]{.384} = (7 + f) + (1 - f) = 8$; &c.

### Ex. 10.

8. (iii) $\sqrt{-64} = 8\sqrt{-1}$;
$\therefore\ \sqrt[4]{-64} = \sqrt{(8\sqrt{-1})} = \sqrt{(0 + 8\sqrt{-1})} = \sqrt{x} + \sqrt{y}$, suppose;
$\therefore\ \sqrt{(0 - 8\sqrt{-1})} = \sqrt{x} - \sqrt{y}$, and $x - y = \sqrt{64} = 8$;
but (119) $x + y = 0$; &c.

### Ex. 11.

4. $2x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} = 5$: mult. by $x^{\frac{1}{2}}$; then $2x - 5x^{\frac{1}{2}} + 2 = 0$;
$\therefore\ x^{\frac{1}{2}} = 2$ or $\frac{1}{2}$, and $x = 4$ or $\frac{1}{4}$.

13. Put $z = \sqrt[4]{(x + 7)}$; $\therefore\ z + \frac{3}{4}z^2 = 5$.

14. Put $z = \sqrt{(7x^2 - 5x + 1)}$; $\therefore\ z^2 - 8z + 7 = 0$, whence $z = 1$ or $7$;
$\therefore\ 7x^2 - 5x + 1 = 1$ or $49$, &c.

15. Put $z = \sqrt{(11x - 2x^2 + 2)}$; then the equation being (when multiplied by 2 and reduced) $2x^2 - 11x + 14z = 42$, we have $z^2 - 14z + 40 = 0$, whence $z$, and then $x$.

16. Put $z = \sqrt{(2x^2 - 7x + 6)}$; then $z^2 + 2z = 0$, and $z = 0$ or $-2$.

18. The equation being $ax + a\sqrt{(x^2 - ax + b^2)} = x^2 + ab$,
put $z = \sqrt{(x^2 - ax + b^2)}$. For 19 and 20, see 15.

21. $(x^2 - 16) + \sqrt{(x^2 - 16)} = 12$; put $\sqrt{(x^2 - 16)} = z$.

22. $\left(\dfrac{1 + x}{1 - x}\right)^{\frac{2}{5}} + \dfrac{3}{16} = \dfrac{(1 - x^2)^{\frac{1}{5}}}{(1 - x)^{\frac{2}{5}}} = \left(\dfrac{1 + x}{1 - x}\right)^{\frac{1}{5}}$; or $z^2 + \frac{3}{16} = z$.

23.* Square and transpose: then
$$x^4 - 6x^2 + 9 = 9x^2 - 42x + 49, \text{ or } x^2 - 3 = \pm(3x - 7).$$

24.* $x^6 - 98x^4 + 2401x^2 = 9x^2 - 126x + 441$; $\therefore\ x\,(x^2 - 49) = \pm 3\,(x - 7)$,
and $x = 7$, or $x^2 - 7x = \pm 3$.

## Ex. **12.**

1. $(x^2+2+x^{-2})+(x+x^{-1})=6$, or $z^2+z-6=0$, whence $z=2$ or $-3$: if $x+x^{-1}=2$, then $x^2+1=2x$, &c.
2. Put $z=\sqrt{(x^2-5x+5)}$; $\therefore$ $5x-x^2=5-z^2$, and (from the equation) $3(5-z^2)+4z=16$, &c.
3. $(x^2-2+x^{-2})+(x-x^{-1})=0$, or $z^2+z=0$, whence $z=0$ or $-1$.
4. Since $x=1$ is a root, divide by $x-1$; then
$x^4+x^3+x^2+x+1=0$, or $x^2+x+1+x^{-1}+x^{-2}=0$,
or $(x^2+2+x^{-2})+(x+x^{-1})-1=z^2+z-1=0$;
hence $z=\frac{1}{2}(-1\pm\sqrt{5})=x+x^{-1}$, and $(-1\pm\sqrt{5})x=2x^2+2$, &c.
5. $x^3+1=p(x+1)$; $\therefore$ $x+1=0$, and $x^2-x+1=p$.
6. One root is evidently $x=1$; div. by $x-1$, we have $2x^2+x+1=0$.
7. See [63 Ex. 2.] 8. $x^3+1+px(x+1)=(x+1)(x^2-x+1+px)=0$.
9. $(n^2+1)x^2-4nax+(n^2+1)(a^2-c^2)=0$, a common quadratic.
10. $(x^4+1)-(x^3+x)-\frac{19}{36}x^2=0$, a recurring equation,
whence $36z^2-36z-91=0$, &c.: see Ex. **12.** 1.
11. $x^4-1=3x(x^2-1)$; $\therefore$ $x^2-1=0$, and $x^2+1=3x$.
12. $x^3-x=2(x+1)$; $\therefore$ $x+1=0$, and $x(x-1)=2$.
13. $x-1=2(1+x^{-\frac{1}{2}})=2x^{-\frac{1}{2}}(x^{\frac{1}{2}}+1)$;
$\therefore$ $x^{\frac{1}{2}}+1=0$ (i), and $x^{\frac{1}{2}}-1=2x^{-\frac{1}{2}}$, or $x-x^{\frac{1}{2}}-2=0$ (ii).
14. $\sqrt[3]{\left(\frac{1+x}{1-x}\right)^2}-1=\sqrt[3]{\frac{1+x}{1-x}}$, or $z^2-1=z$, if $z=\sqrt[3]{\frac{1+x}{1-x}}$.
15. $4(n^2x+nax^2)=\{(3n-a)x-n\}^2=(3n-a)^2x^2-2(3n-a)nx+n^2$;
$\therefore$ $(9n^2-10na+a^2)x^2-2n(5n-a)x+n^2=0$,
which solve as a quadratic by (128), observing that
$$9n^2-10na+a^2=(9n-a)(n-a).$$
16.* $a(\sqrt{x}+\sqrt{b})=(x-b)(\sqrt{x}-\sqrt{b})=(\sqrt{x}+\sqrt{b})(\sqrt{x}-\sqrt{b})^2$;
$$\therefore\ \sqrt{x}+\sqrt{b}=0, \text{ and } \sqrt{x}-\sqrt{b}=\pm\sqrt{a}.$$
17. $x^3-1=2px(x-1)$; $\therefore$ $x-1=0$, and $x^2+x+1=2px$,
$$\text{or } x^2-(2p-1)x+1=0.$$
18. $x^4+1-3(x^3-x)-\frac{4}{9}x^2=0$; or $9(x^2+x^{-2})-27(x-x^{-1})-4=0$;
$\therefore$ $9z^2-27z+14=0$, if $z=x-x^{-1}$, &c., as in Ex. **12.** 1.
19. $x^3-\frac{4}{9}=\frac{13}{9}x$, whence $9x^3-13x=4$, or $9x^3-9x=4x+4$;
$\therefore$ $9x(x^2-1)=4(x+1)$, whence $x+1=0$ and $9x(x-1)=4$, &c.
20. $x^{12}-1=(x^6+1)(x^6-1)=(x^2+1)(x^4-x^2+1)(x^2-1)(x^4+x^2+1)=0$;
$\therefore$ $x^2+1=0$, $x=\pm\sqrt{-1}$; $x^2-1=0$, $x=\pm 1$; $x^4-x^2+1=0$,
$x^2=\frac{1}{2}(1\pm\sqrt{-3})=\frac{1}{4}(2\pm 2\sqrt{-3})$, and $x=\pm\frac{1}{2}(\sqrt{3}\pm\sqrt{-1})$;
$x^4+x^2+1=0$, $x^2=\frac{1}{4}(-2\pm 2\sqrt{-3})$, and $x=\pm\frac{1}{2}(1\pm\sqrt{-3})$.

21. $(x^4+1)-\frac{1}{2}(x^3-x)-\frac{7}{2}x^2=0$: see Ex. **12.** 18.

22.* $x+a=(x^{\frac{1}{3}}+a^{\frac{1}{3}})^3-3a^{\frac{1}{3}}x^{\frac{1}{3}}(x^{\frac{1}{3}}+a^{\frac{1}{3}})=b-3a^{\frac{1}{3}}x^{\frac{1}{3}}b^{\frac{1}{3}}$,

or $(x^{\frac{1}{3}}+a^{\frac{1}{3}})^3-b=3a^{\frac{1}{3}}x^{\frac{1}{3}}\{(x^{\frac{1}{3}}+a^{\frac{1}{3}})-b^{\frac{1}{3}}\}$; $\therefore\ (x^{\frac{1}{3}}+a)^{\frac{1}{3}}-b^{\frac{1}{3}}=0$,

and $(x^{\frac{1}{3}}+a^{\frac{1}{3}})^2+(x^{\frac{1}{3}}+a^{\frac{1}{3}})\,b^{\frac{1}{3}}+b^{\frac{2}{3}}=3a^{\frac{1}{3}}x^{\frac{1}{3}}$,

or $x^{\frac{2}{3}}-(a^{\frac{1}{3}}-b^{\frac{1}{3}})\,x^{\frac{1}{3}}+(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})=0$, a common quadratic.

23. $(x^2+9)+\sqrt{(x^2+9)}+\frac{1}{4}=x^4-x^2+\frac{1}{4}$, or $\sqrt{(x^2+9)}+\frac{1}{2}=\pm(x^2-\frac{1}{2})$;
$\therefore$ (i) $\sqrt{(x^2+9)}=x^2-1$, and $x^2+9=x^4-2x^2+1$, &c.;
(ii) $\sqrt{(x^2+9)}=-x^2$, and $x^2+9=x^4$, &c.

24. $4x^2+12x\sqrt{(1+x)}+9(1+x)=36(1+x)$;
$\therefore\ 2x+3\sqrt{(1+x)}=\pm 6\sqrt{(1+x)}$;
$\therefore$ (i) $2x=3\sqrt{(1+x)}$, $4x^2=9(1+x)$;
(ii) $2x=-9\sqrt{(1+x)}$, $4x^2=81(1+x)$.

25. $(x+1)^2-8(x+1)\sqrt{x}+16x=0$; $\therefore\ (x+1)-4\sqrt{x}=0$.

26. By (85) $\dfrac{\sqrt{x}}{\sqrt{(x-a)}}=\dfrac{n^2a+(x-a)}{n^2a-(x-a)}$, and $\therefore$ also $\dfrac{x}{a}=\dfrac{\{n^2a+(x-a)\}^2}{4n^2a(x-a)}$;
whence $(4n^2-1)x^2-2(3n^2-1)ax-(n^2-1)^2a^2=0$.

27.* $(a-1)(x^4+1)-4a(x^3+1)+6ax^2=0$,
or $(a-1)(x^2+x^{-2})-4a(x+x^{-1})+6a=0$,
whence $(a-1)z^2-4az+(4a+2)=0$, if $z=x+x^{-1}$.

28. By (85), $\dfrac{a+x}{\sqrt{(a^2-x^2)}}=\dfrac{c+x}{c-x}=\sqrt{\dfrac{a+x}{a-x}}$;

$\therefore\ \dfrac{a+x}{a-x}=\left(\dfrac{c+x}{c-x}\right)^2$, and again by (85), $\dfrac{x}{a}=\dfrac{2cx}{c^2+x^2}$; &c.

29. By (85), $\dfrac{a-\sqrt{(a^2-ax)}}{a}=\left(\dfrac{1-c}{1+c}\right)^2=1-\sqrt{(1-\dfrac{x}{a})}$; &c.

30. By (85), $\dfrac{ax+1}{\sqrt{(a^2x^2-1)}}=\sqrt{\dfrac{ax+1}{ax-1}}=\dfrac{b^2x+2}{b^2x-2}$; $\therefore\ \dfrac{ax}{1}=\dfrac{b^4x^2+4}{4b^2x}$.

31. From (1),
$x^4+y^4+4xy(x^2+y^2)+6x^2y^2=625=97+4xy(25-2xy)+6x^2y^2$;
whence $xy$, and then $x$ and $y$ from (1).

32. See Ex. **12.** 31.

33. Divide (1) by (2), and subtract the quotient from the fourth power of (2); then $5xy(x^2+xy+y^2)=b^4-\dfrac{a^5}{b}=5xy(b^2-xy)$, whence $xy$, &c.

34. From (1) $\frac{2-x^2-y^2}{(1-y)(1-x)}=\frac{9}{13}$, from (2) $\frac{2-x^2-y^2}{(1+y)(1-x)}=\frac{4}{13}$;

$$\therefore \frac{1+y}{1-y}=\frac{9}{4},\quad y=\frac{5}{13};$$

and (1) becomes, by putting $y=\frac{5}{13}$, $169x^2-72x-241=0$. Also (1) and (2) are satisfied by putting together $2-x^2-y^2=0$, and $1-x=0$; whence $x=1$, $y=\pm 1$.

35. From (1) $x^2y^2=a^2(x^2+2xy+y^2)=a^2\left(\frac{x^2y^2}{b^2}+2xy\right)$, whence $xy$:

(i) $xy=0$ will plainly require in (1) and (2) $x=0$, $y=0$;

(ii) $xy=\frac{2a^2b^2}{b^2-a^2}$, gives in (1) $x+y=\frac{2ab^2}{b^2-a^2}$, &c.

36. $1+\frac{x+y}{3x}=2\sqrt{\frac{x+y}{3x}}$, or $1+z^2=2z$, whence $z=1$, &c.

37. In (1) put $y=vx$; then $4v^2-7v+3=0$, and $v=1$ or $\frac{3}{4}$: in (2) put (i) $y=x$, and (ii) $y=\frac{3}{4}x$.

38. In (1) and (2) put $y=vx$; then

$x^2(1+v^{\frac{2}{3}})=208$, $x^2(v^2+v^{\frac{4}{3}})=1053$; and $\frac{1053}{208}=v^{\frac{4}{3}}=\frac{81}{16}$;

$\therefore v=\frac{27}{8}$, $x^2=208\div(1+v^{\frac{2}{3}})=64$, &c.

39. $\frac{x^2}{y^2}+\frac{2x}{\sqrt{y}}+\sqrt{y}=20-y-\frac{x}{y}$, or $\left(\frac{x^2}{y^2}+\frac{x}{y}+\frac{1}{4}\right)+2\sqrt{y}\left(\frac{x}{y}+\frac{1}{2}\right)+y$

$=20+\frac{1}{4}=\frac{81}{4}$; $\therefore \left(\frac{x}{y}+\frac{1}{2}\right)+\sqrt{y}=\pm\frac{9}{2}$, in which put $x=4y-8$

from (2): then $\frac{4y-8}{y}+\sqrt{y}=4$ or $-5$; (i) gives $y^{\frac{1}{2}}=2$, $y=4$,

$x=8$; (ii) gives $y^{\frac{3}{2}}+9y-8=0$, of which a root is $y^{\frac{1}{2}}=-1$,

$\therefore y=1$, $x=-4$; and the other factor is $y+8y^{\frac{1}{2}}-8=0$, &c.

40. From (1) $\frac{x^2-y^2}{xy}=\frac{x+y}{x^2+y^2}$, from (2) $\frac{x^4-y^4}{x^2y^2}=\frac{x-y}{y^2}$;

$\therefore$ (2) $\div$ (1) gives $\frac{x^2+y^2}{xy}=\frac{(x-y)(x^2+y^2)}{(x+y)y^2}$, or $\frac{y}{x}=\frac{x-y}{x+y}$,

whence $x^2-y^2=2xy$, and $x^2-2xy+y^2=2y^2$; $\therefore x=y(1\pm\sqrt{2})$;

$\therefore$ (1) gives $2=\frac{x+y}{x^2+y^2}$, or $2(x^2+y^2)=x+y$;

$\therefore 2y^2(4\pm2\sqrt{2})=y(2\pm\sqrt{2})$; $\therefore y=0$ or $\frac{1}{4}$, $x=0$ or $\frac{1}{4}(1\pm\sqrt{2})$.

41. From (1) $x^2 + xy + y^2 = \{x + \sqrt{(xy)} + y\}\{x - \sqrt{(xy)} + y\} = a^2$;

$$\text{but (2) } x + \sqrt{(xy)} + y = b; \ \therefore\ x - \sqrt{(xy)} + y = \frac{a^2}{b};$$

adding and subtracting these two equations, we get $x + y$ and $\sqrt{(xy)}$.

42. (1)×(2) gives $(xy)^{m+n} = a^{2m}b^{2n}cd$, and $\therefore\ xy = (a^{2m}b^{2n}cd)^{\frac{1}{m+n}}$;

(1)÷(2) gives $(xy^{-1})^{m-n} = cd^{-1}$, and $\therefore\ xy^{-1} = (cd^{-1})^{\frac{1}{m-n}}$;

multiply together these results for $x$, and divide for $y$.

43. $\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \sqrt{x}$; $\therefore\ \sqrt{\dfrac{x}{y}} = \dfrac{1 + \sqrt{x}}{1 - \sqrt{x}}$, or $\dfrac{x}{y} = \left(\dfrac{1 + \sqrt{x}}{1 - \sqrt{x}}\right)^2$, whence

$\dfrac{x - y}{x + y} = \dfrac{2\sqrt{x}}{1 + x} = \dfrac{1}{\sqrt{2}}$ from (2); $\therefore\ x - 2\sqrt{(2x)} + 1 = 0$, &c.

44. Put $y = xz$; then (1)÷(2) gives $z^4 = \dfrac{az - b}{a - bz}$, or $az(z^3 - 1) = b(z^5 - 1)$;

$\therefore\ z = 1$, and, dividing by $z - 1$, we get

$$az(z^2 + z + 1) = b(z^4 + z^3 + z^2 + z + 1),$$

a recurring equation, which may be solved as in [63 Ex. 3]; then from (2), $x = a - bx$, $y = zx$.

45. Square the equation: then, by reason of (2), the middle terms will disappear, and we get

$y^2(a-x)(x-b) + x^2(a-y)(b-y) = 4\{b^2(a-x)(a-y) + a^2(x-b)(b-y)\}$,

whence $(a-x)\{y^2(x-b) - 4b^2(a-y)\} = (b-y)\{4a^2(x-b) - x^2(a-y)\}$,

or $(a-x)\{4ab(y-b) - by^2 + 4b^2y\} = (b-y)\{4a^2x - ax^2 + 4ab(x-a)\}$;

$$\therefore\ (a - x)\left(-\frac{4ab^2y}{x} + 4b^2y\right) = (b - y)\left(4a^2x - \frac{4a^2bx}{y}\right),$$

or $b^2y(a - x)(x - a) = a^2x(b - y)(y - b)$; $\therefore\ \dfrac{x - a}{y - b} = \pm\dfrac{a}{b}$, where substitute for $y$ from (2).

46. From the square of (1) subtracting (2) we get

$2(xy + xz + yz) = 112 = 2(xy + y^2 + yz) = 2y(x + y + z) = 28y$;

$\therefore\ y = 4$, whence $x + z = 10$, $x^2 + z^2 = 68$, &c.

47. (1) × (2) × (3) gives $x^4y^4z^4 = a^3b^3c^3xyz$, or $xyz = abc$;

$\therefore$ (1) gives $\dfrac{a^2b^2c^2}{z^2} = c^3z$, &c.

48. See (46).

49. $(871 - y^2)^2 + y^2 + (877 - y^2)^2 = 3037$,

or $2y^4 - 3495y^2 + 1524733 = 0$; &c.

50. See (46).

## Ex. 13.

N.B. Observe that *zero*-values of $x$ and $y$ are not reckoned.

1. $3x + 5y = 100 - 7z$: take $z$ from 1 to 13 (not 14, for then we should have $c < (a + b)$;) [72]

$$\text{if } z = 1, \text{ then } 3x + 5y = 93, \text{ or } x + \tfrac{5}{3}y = 31,$$

whence (since $x$ and 31 are integers) we must have $y = 3t$; and $\therefore x = 31 - 5t$, which has 6 solutions: so giving $z$ the successive values, 2, 3, &c., 13, the resulting equations will have 6, 5, 4, 4, 4, 3, 3, 2, 1, 2, 1, 0, solutions respectively, and the sum of all the solutions is therefore 41.

$5x + 7y = 144 - 11z$: take $z$ from 1 to 12; the number of solutions will be $3 + 4 + 3 + 2 + 3 + 2 + 2 + 1 + 1 + 1 + 1 + 1 = 24$.

$17x + 19y = 200 - 21z$: take $z$ from 1 to 7; the number of solutions will be $0 + 0 + 0 + 0 + 0 + 1 + 1 = 2$.

[This equation may be best treated by putting $x'$ for $x + y$; thus, if $z = 1$, then $17x + 19y = 179 = 17x' + 2y$, whence (as usual) $x'$ or $x + y = 2t + 1$, $y = 81 - 17t$, and $\therefore x = 19t - 80$, which give no positive integral *simultaneous* values for both $x$ and $y$.]

2. Expressing all in *sixpences*, we have $x + y + z = 30$ (i) and $7x + 9y + 13z = 360$ (ii), whence $2y + 6z = 360 - 7(x + y + z) = 150$; $\therefore y = 75 - 3z$, whence $z < 25$; and also $y + z = 75 - 2z < 30$ by (i); whence $2z > 45$, or $z > 22$; hence the solutions are $z = 23$, $y = 6$, $x = 1$, or $z = 24$, $y = 3$, $x = 3$.

3. Expressing in *fourpences*, $12x + 9y + z = 120$, or $9y + z = 120 - 12x$, where we must evidently take the greatest possible value for $x$, viz. 9; and then $9y + z = 12$, where $y = 1$, $z = 3$, and $x + y + z = 13$.

4. Let the number of gallons required be $100x$, $100y$, $100z$;

$$\text{then (1) } x + y + z = 9, \quad \text{(2) } 12x + 15y + 18z = 153;$$

$\therefore$ (2) $- 12 \times$ (1) gives $3y + 6z = 153 - 12 \times 9 = 45$, and $y = 15 - 2z$, whence $z < 8$; also $y + z = 15 - z$, and by (1) $y + z < 9$; $\therefore z > 15 - 9 > 6$; $\therefore z = 7$, $y = 1$, $x = 1$.

5. (1) $x + y + z = 50$, (2) $7x + 12y + 17z = 750$; $\therefore$ (2) $- 7 \times$ (1) gives $5y + 10z = 750 - 350 = 400$, and $y = 80 - 2z$, whence $z < 40$; also $y + z = 80 - z$, and by (1) $< 50$; $\therefore z > 80 - 50 > 30$, and $z$ may range from 31 to 39.

6. $15x + 16y + 20z = (x + y + z + 10)\, 11\frac{1}{3}$, or $11x + 14y = 340 - 26z$, whence $z < \frac{340}{26} < 14$: but [72] we see at once that it is useless to take $z > 12$, and, on trial, it will be found that 10 is the highest value that can be taken for $z$, when $11x + 14y = 80$, whence $x = 6$, $y = 1$.

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### Ex. **15.**

1. Mult. by 12: then $3(x+2)+4x<6(x-4)+36$ and $>6(x+1)+4$, which gives $x<6$ and $>4$; $\therefore\ x=5$.

2. $\frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab}$, which is $>2$, since $a^2 + b^2 > 2ab$.

3. (ii) $\frac{a}{b^2} + \frac{b}{a^2} > \frac{1}{b} + \frac{1}{a}$, if $\frac{a^3 + b^3}{a^2b^2} > \frac{a + b}{ab}$, or $a^3 + b^3 > ab(a + b)$,

$$\text{or } a^2 - ab + b^2 > ab, \quad \text{or } a^2 + b^2 > 2ab.$$

5. $x^2y^2 = a^2c^2 + b^2d^2 + (a^2d^2 + b^2c^2)$; but $a^2d^2 + b^2c^2 > 2ad.bc > 2acbd$;

$$\therefore\ x^2y^2 > a^2c^2 + 2acbd + b^2d^2, \text{ \&c.}$$

6. $\frac{a^4 - b^4}{a - b} = a^3 + a^2b + ab^2 + b^3 < a^3 + a^3 + a^3 + a^3 < 4a^3$;

$$\therefore\ a^4 - b^4 < 4a^3(a - b).$$

7. $a^2 + b^2 > 2ab$, $a^2 + c^2 > 2ac$, $b^2 + c^2 > 2bc$: add these results.

8. By [74 Ex. 2] $a^3+b^3>a^2b+ab^2$, $b^3+c^3>b^2c+bc^2$, $a^3+c^3>a^2c+ac^2$: add these results.

9. $a^2+b^2>2ab$; $\therefore\ a^2c+b^2c>2abc$; so $a^2b+ab^2>2abc$, $a^2c+b^2c>2abc$; add these results.

10. $(a+b+c)^3 = (a^3+b^3+c^3) + 3(a^2b+a^2c+\text{\&c.})+6abc > \frac{1}{2}(a^2b+\text{\&c.}) + 3(a^2b + \text{\&c.}) + 6abc > 21abc + 6abc$, by last two Examples.

11. $(1-a)^4$ is *positive*, whence $1+6a^2+a^4>4(a+a^3)$; but $2a<1+a^2$, or $2a^2 < a + a^3$; $\therefore$ *a fortiori*, $1 + 4a^2 + 2a^4 > 3(a + a^3)$.

12. $(a+b-c)(a+c-b)=a^2-(b-c)^2<a^2$; so $(b+a-c)(b+c-a)<b^2$, $(c+a-b)(c+b-a)<c^2$; $\therefore\ (a+b-c)^2(a+c-b)^2(b+c-a)^2<a^2b^2c^2$.

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### Ex. **17.**

4. Let $A = mB$, $C = xB$, $D = yB$; then, if $AC = nD^2$ and $B^2 = pCD$, we have (i) $mx = ny^2$, (ii) $1 = pxy$,

whence $x = \sqrt[3]{\frac{n}{mp^2}}$, $y = \sqrt[3]{\frac{m}{np}}$, that is, both are *constants*;

$$\therefore\ \frac{(A + B + C)^4}{ABCD} = \frac{(m + 1 + x)^4}{mxy}, \text{ a \textit{constant} quantity.}$$

### Ex. 18.

5. As in [88 Ex.] $n=10$, or the 20 men must be combined 10 together. Now leave out the particular man; and form Combns of the other 19 men, 9 together, into each of which introduce the man left out, and we have the Combns required.

6. If all the numbers on the two dice, ($A$ and $B$,) were different, the n° of different throws would be $m(m+n)$; but here the throws will be of the form 1 and 1, 1 and 2, 2 and 1, 1 and 3, 3 and 1, &c., and it is plain that there will be $m(m-1)$ pairs of duplicate throws: (thus the 1 of $A$ may be taken with the 2, 3, 4, &c. $m$, of $B$, the 2 of $A$ with 1, 3, 4, &c. $m$, of $B$, the 3 of $A$ with 1, 2, 4, &c. $m$, of $B$, and so on, every one of which throws will plainly be repeated;) hence the whole n° of different throws will be $m(m+n)-\frac{1}{2}m(m-1)=mn+\frac{1}{2}m(m+1)$.

7. Instead of figures 1, 2, 3, &c. let us use letters $a$, $b$, $c$, &c., and suppose $A$ to be marked with $a, b, c$, $B$ with $a, b, c, d, e, f$, $C$ with $a$, $b$, $c$, &c. $g$, $h$, $D$ with $a$, $b$, $c$, &c. $k$, $l$, $m$, $n$. Now the number of different throws will plainly be the number of *different* terms in the product

$$(a+b+c)\ (a+b+c+d+e+f)\ (a+b+\&c+g+h)\ (a+b+\&c+l+m+n)$$
$$=v(v+x)\ (v+x+y)\ (v+x+y+z)\ \text{suppose}$$
$$=v\{(v+x)^3+(v+x)^2\ (2y+z)+(v+x)\ y(y+z)\}$$
$$=v^4+3v^3x+3v^2x^2+vx^3+2v^3y+4v^2xy+2vx^2y+v^3z+2v^2xz$$
$$+vx^2z+v^2y^2+v^2yz+vxy^2+vxyz.$$

Now $v^4=(a+b+c)^4$, in which the number of different terms is $\frac{3.4.5.6}{1.2.3.4}=15$; again (in $3v^3x$ or) in $v^3x=(a+b+c)^3(d+e+f)$ it is $\frac{3.4.5}{1.2.3}\times 3=30$; proceeding similarly with the rest, (observing that $y=g+h$, $z=k+l+m+n$) we have the whole number of different terms in the product

$$15+30+36+30+20+36+36+40+72+72+18+48+27+72=552.$$

8. (i) Any *three* of the $n-p$ points may be taken together, thus forming $\frac{1}{6}(n-p)(n-p-1)(n-p-2)$ triangles; (ii) any *two* of the $n-p$ may be taken with any *one* of the $p$ points, thus forming $\frac{1}{2}(n-p)(n-p-1)\times p$ triangles; (iii) any *one* of the $n-p$ may be taken with any *two* of the $p$ points, thus forming $(n-p)\times\frac{1}{2}p(p-1)$ triangles: the sum of these three, when reduced, gives the Answer.

## Ex. 25.

3. $\log 20000 = 4.3010300$, $\log 20001 = 4.3010517$; $\therefore\ D = 117$, &c.

4. $\log 31000 = 4.4913617$, $\log 31001 = 4.4913757$; $\therefore\ D = 140$, and (i) for 31000.23, $\delta = \frac{23}{100} \times 140 = 23 \times 1.4 = 32$; (ii) for $31000\frac{1}{4}$, $\delta = \frac{1}{4} \times 140 = 35$; (iii) $31\frac{3}{4000} = 31.00075$, and for 31000.75 or $31000\frac{3}{4}$, $\delta = \frac{3}{4} \times 140 = 105$: in each case the value of $\delta$ is to be added to the *mantissa* for 31, and the Char^c^. determined by inspection.

5. $.1839883 + .1706278 + .1558123 + .1426420 = .6530704$, from which, taking the lowest of the two given mantissæ, we have $\delta = 27$; also $D =$ diff. of given mantissæ $= 97$, for which we may construct a Table of Pro. Parts; or thus, by [112],

$$d = \frac{\delta}{D} = \frac{27}{97} = .27.$$

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## Ex. 26.

3. $x = \dfrac{\log 800}{\log 5} = \dfrac{2.9030900}{.6989700} = 4.153.$

4. $a^{\frac{nx}{m}} = b^{nx-a}$; $\therefore\ \dfrac{nx}{m} \log a = (nx - a) \log b$, or

$$x\,(mn \log b - n \log a) = ma \log b.$$

5. (1) gives $x \log a + y \log b = \log c$, from which and from (2) find $x$ and $y$.

6. $(a+b)^2\,(a+b)^{x-1} \times (a-b)^{x-1} = (a-b)^{2x}$; $\therefore\ (a+b)^{x+1} = (a-b)^{x+1}$, whence $x + 1 = 0$.

7. $x^{2x} - 1 = 3\,(x^x + 1)$, or $x^{2x} - 3x^x - 4 = 0$, whence $x^x = 4$ or $-1$, and $x$ manifestly $= 2$ or $-1$.

8. $2x\,(\log 5 - \log 2) + (3x - 5) \log 5 = (x + 1)\,(\log 5 - 2 \log 2)$
$+ (2 - x)\,(3 \log 2 - \log 5)$, whence

$$x\,(3 \log 2 + 3 \log 5) = 4 \log 5 + 4 \log 2;\ \therefore\ x = 1\tfrac{1}{3}.$$

9. From (1) $x^{by} = y^{bx} = (y^b)^x = x^{ax}$ from (2); $\therefore\ by = ax$: whence substitute for $y$ in (2).

10. Dividing by $a^{2x}$, we have $a^{2x} + 1 = a^{4x}$, or $a^{4x} - a^{2x} - 1 = 0$, &c.

11. $a^{2x} + 1 = ba^x$, or $a^{2x} - ba^x + 1 = 0$, &c.

12. Put $z = \frac{1}{2}(\sqrt[4]{x} + \sqrt[4]{y})$; $\therefore\ z \log x = \frac{4}{3} \log y$, $z \log y = \frac{1}{3} \log x$, whence $\log y = \frac{1}{2} \log x$, and $\therefore\ z = \frac{2}{3}$;

hence (ii) $y^{\frac{2}{3}} = x^{\frac{1}{3}}$, or $y^2 = x$;

$\therefore\ z = \frac{1}{2}(x^{\frac{1}{4}} + y^{\frac{1}{4}}) = \frac{1}{2}(x^{\frac{1}{4}} + x^{\frac{1}{8}}) = \frac{2}{3}$, whence $x^{\frac{1}{8}} = \frac{1}{6}(-3 \pm \sqrt{57})$.

### Ex. 28.

2. The question is the same as if the sum had been at Interest for 20 yrs at $2\frac{1}{2}$ per cent. per ann.; $\therefore M = PR^n = 50\,(1.025)^{20}$, and $\log M = \log 50 + 20 \log (1.025)$: observe that $1025 = 25 \times 41$, and see the logs on in [107].

4. $M = PR^n = £\frac{1}{960} \times (1.06)^{200}$: $\therefore \log M = 200 \log (1.06) - \log 960$.

5. By [147] $M = A\,.\,\dfrac{R^n - 1}{R - 1} = 1000 \times \dfrac{(1.04)^3 - 1}{.04} = \dfrac{100000}{4} \times .124864$ $= 3121.6$.

6. By (198) $V = \dfrac{M}{R^n} = \dfrac{3121.6}{1.124864} = 2775.09$.

10. The gain by Compound Interest in four quarters is

$$P\{R^n - 1\} = P\left\{\left(1 + \frac{1\frac{1}{4}}{100}\right)^4 - 1\right\} = P\left\{\left(\frac{81}{80}\right)^4 - 1\right\};$$

the loss by Simple Interest in one year is

$$P\,\frac{3\frac{1}{2}}{100};\ \therefore 441 = P\left\{\left(\frac{81}{80}\right)^4 - \frac{207}{200}\right\},\ \text{whence}$$

$49 \times 80^4 = P\{9 \times 81^3 - 23 \times 204800\} = 72569P$, or $80^4 = 1481P$.

11. By [148] $P$ is equivalent to an extra rent $\dfrac{Pr}{1 - R^{-n}}$, and the new fine ($x$) to a rent $\dfrac{xr}{1 - R^{-(m+n)}}$, which must equal the former.

12. The present value of $A$'s legacy $= \dfrac{A}{r}(1 - R^{-2}) = \dfrac{25}{16}$ of $B$'s

$$= \frac{25}{16} \times \frac{A}{r}(R^{-2} - R^{-4}) = \frac{25}{16} \cdot \frac{A}{r} \cdot R^{-2}(1 - R^{-2});$$

$\therefore R^2 = \frac{25}{16}$, $r = \frac{1}{4}$, and $A$'s legacy is worth $\dfrac{A}{\frac{1}{4}}\left(1 - \dfrac{16}{25}\right) = £1440$, and $B$'s $= \frac{16}{25}$ of $£1440 = £921\ 12s.$

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### Ex. 32.

1. Find the converging fractions to $\frac{314159}{100000}$. In (i) the error (in excess) $< \dfrac{1}{7 \times 106} < .002$, and $\therefore \frac{22}{7}$ (= 3.142 &c.) may be relied on for *two* places of decimals: in (ii) the error (in defect) $< \dfrac{1}{106 \times 113} < .000084$; and $\therefore \frac{333}{106}$ (= 3.14150 &c.) may be relied on to *four* places: in (iii), since the denominator of the next convergent is 2931, the error (in excess)

$< \frac{1}{113 \times 2931} < .00000302$, and $\therefore \frac{355}{113}$ (= 3.141592 &c.), although really correct to *five* places, can only be relied on to *four*, since we have no reason to be sure that the error when taken from 3.141592 &c., will not change the *fifth* decimal figure, 9.

6. (i) $\sqrt{2} = 1 + \frac{1}{2+}\;\frac{1}{2 + \&c.} = 1,\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70},\ \frac{239}{169}$, &c.;

(ii) $\sqrt{3} = 1 + \frac{1}{1+}\;\frac{1}{2 + \&c.} = 1,\ 2,\ \frac{5}{3},\ \frac{7}{4},\ \frac{19}{11},\ \frac{26}{15},\ \frac{71}{41},\ \frac{97}{56},\ \frac{265}{153}$, &c.

the limit of the error being in

(i) $< \frac{1}{70 \times 169} = .000083$ &c., in (ii) $< \frac{1}{56 \times 153} = .00011$ &c.

7. Let $x$, $y$, be the times respectively by $M$ and $V$ for one revolution; then $x = 88$, $y = 225$; $\therefore y = \frac{225}{88}x = \frac{23}{9}x$, nearly, as we find by converting $\frac{225}{88}$ to a continued fraction: $\therefore 9y = 23x$, or the time taken by $V$ for 9 revolutions is (nearly) the same as that taken by $M$ for 23 revolutions.

8. Let the div$^n$ $x$ of the first scale coincide with $y$ of the second; then (the number of divisions in a given space being inversely as the space between each) $x : y :: 1.06577 : 1$, or

$x = \frac{106577}{100000}y = \frac{16}{15}y$ nearly, or $\frac{65}{61}y$, or $\frac{81}{76}y$, &c.:

$\therefore$ if $y = 15,\ 61,\ 76$, &c., we have $x = 16,\ 65,\ 81$, &c.

9. (i) $x = \sqrt{\frac{7}{3}} = \frac{\sqrt{21}}{3} = 1 + \frac{\sqrt{21} - 3}{3} = 1 + \frac{4}{\sqrt{21} + 3}$, &c. as in [154]

$$= 1 + \frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{8+}\;\frac{1}{1+}\;\frac{1}{1 + \&c.} = 1,\ 2,\ \frac{3}{2},\ \frac{26}{17},\ \frac{29}{19},\ \frac{55}{36} = 1.52 \ \&c$$

(ii) $x = \frac{5 \pm \sqrt{13}}{6}$, to be treated as in [162 Ex. 1.]

0. (i) $x - p = \frac{1}{q+}\;\frac{1}{r + (x - p)} = \frac{r + (x - p)}{qr + 1 + q\,(x - p)}$, whence $x - p =$ &c.:

(ii) Let $y = q + \frac{1}{r+}\;\frac{1}{s + \&c.}$; find $y$ as in (i), and then $x = p + \frac{1}{y}$.

11. (i) $x = \sqrt{(3 + x)}$, or $x^2 - x - 3 = 0$;

$\therefore x = \frac{\sqrt{13} + 1}{2} = 2 + \frac{1}{3+}\;\frac{1}{3 + \&c.} = 2,\ \frac{7}{3},\ \frac{23}{10},\ \frac{76}{33}$.

(ii) Since $x$ must be negative and $< 1$, put $-\frac{1}{y}$ for $x$; then $2^y = 3$; ∴ by [162 Ex. 2] $y = \frac{8}{5}$, and $x = -\frac{5}{8}$.

(iii) $x = \frac{1}{y} = \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{3+\&c.} = 1,\ \frac{1}{2},\ \frac{3}{5},\ \frac{10}{17}$.

12. $x = \sqrt{(a + \frac{b}{x})}$, or $x^3 - ax - b = 0$.

## Ex. 33.

7. The solution $x = 398$, $y = 60$, is obtained by the method in [166 Cor. 2].

## Ex. 39.

4. Let $2m = p\,(2n + 1)$: then the factor $p$ must be divisible by 2, that is, must be of the form $2p'$; and ∴ $m = p'\,(2n + 1)$.

5. (i) $(2m+1)^2 - (2n+1)^2 = 4\,(m^2 - n^2 + m - n) = 4\,(m - n)\,(m + n + 1)$, where, if $m$ and $n$ are *both* even or odd, $m - n$ is div. by 2, or, if one be even and the other odd, $m + n + 1$ is so divisible.

6. (i) $2m\,(4m^2 - 4) = 8\,(m - 1)\,m\,(m + 1)$, where apply [Ex. **39** 3]: (ii) by [Ex. **39** 3], $n\,(n^2 - 1)$ is div. by 6; besides which, if $n = 2m + 1$, then $(2m + 1)\,(4m^2 + 4m) = 4m\,(m + 1)\,(2m + 1)$ is divisible by 8, since $m$ or $m + 1$ must be even.

7. (i) Every even square must be of the form $(4m)^2$ or $(4m + 2)^2$, the last of which $= 4\,\{4\,(m^2 + m) + 1\}$.
   (ii) $(4m + 1)^2 = 8\,(2m^2 + m) + 1$.

8. Take the different forms to the modulus 6, viz. $6p$, $6p \pm 1$, $6p \pm 2$, $6p \pm 3$, and prove the statement in each case.

9. Let the *odd* square $= 8m + 1$, and the *sum* $= 8n + 1$; then the *even* square $= 8\,(n - m)$, and must therefore [Ex. **39** 7] be of the form $16p$, since it cannot be of the form $4\,(4p + 1)$.

10. That is, $(2m + 1)\,(2n + 1) = (m + n + 1)^2 - (m - n)^2$, since the middle number between them is $\frac{1}{2}\{(2m + 1) + (2n + 1)\}$.

## Ex. 40.

1. Let $N = a + b$, and be (if possible) prime to $a$, but $= mb$; then $a = (m - 1)\,b$, that is, $b$ is a factor of $a$, and ∴ $N = mb$ contains a factor common to $a$, and cannot be prime to it.

2. By Fermat's Theorem, $a^{m-1} - 1 = mp$, $b^{m-1} - 1 = mq$;
$$\therefore\ a^{m-1} - b^{m-1} = m(p - q).$$

3. (i) Take the forms to modulus 4: (ii) apply [197 iii].

4. (i) Any number prime to 4 must be of the form $4p \pm 1$: then $(4p \pm 1)^2 =$ &c.: (ii) By [Ex. **40** 3] every cube number is of the form $4p$ or $4p \pm 1$; $\therefore$ &c.

5. (i) Apply [197 iv]: (ii) apply [197 iii].

6. By Fermat's Theorem, the quantities $1^{n-1}$, $2^{n-1}$, &c. are of the form $np_1 + 1$, $np_2 + 1$, &c.; $\therefore$ their sum is of the form
$$n(p_1 + p_2 + \&c.) + n - 1 \text{ or } np - 1.$$

7. By (Ex. **39** 5) the diff. is div. by 8, and by [197 iii], each square being of the form $3p + 1$, their diff. is also div. by 3.

8. By [197 iii] $N^4 - 1$ is div. by 5, and also $= (N^2 - 1)(N^2 + 1)$, of which by [Ex **39** 5] $N^2 - 1$ is div. by 8, and ($N$ being odd) $N^2 + 1$ is div. by 2.

### Ex. **42.**

6. Put $2x^2 + 2 = 4 + 2(x - 1)(x + 1) = \{2 - \frac{m}{n}(x - 1)\}^2$.

### Ex. **43.**

4. By [204] put $2 + y = z^2$; then $3 + 2y = 2z^2 - 1 = \square$, which has a solution $z = 1$; put $z = v + 1$; then $2v^2 + 4v + 1 = (\frac{m}{n} v + 1)^2$, whence $v$, and then $z = \dfrac{m^2 - 2mn + 2n^2}{m^2 - 2n^2}$: if $m = 3$, $n = 2$, then $z = 5$, $y = 23$, $x = \frac{1}{23}$.

### Ex. **44.**

1. Let $x + 1 = p^2$; $\therefore\ x - 1 = p^2 - 2 = \square = \left(p - \dfrac{m}{n}\right)^2$, whence $p$, &c.

2. Here $\dfrac{x(x + 1)}{2} = \square = \dfrac{m^2}{n^2} x^2$ suppose, whence $x$.

3. $x^2 - y^2 = a$, or $x^2 - a = \square = \left(x - \dfrac{m}{n}\right)^2$, whence $x$ and $y$.

4. $x + y = m^2$, $x - y = n^3$; $\therefore\ x = \frac{1}{2}(m^2 + n^3)$, $y = \frac{1}{2}(m^2 - n^3)$.

5. $y^2 + xy = \square = p^2y^2$ suppose;
$$\therefore\ x = (p^2 - 1)y, \text{ and } x^2 + xy = (p^2 - 1)p^2y^2 = \square;$$
$$\therefore\ p^2 - 1 = \square = \left(p - \frac{m}{n}\right)^2, \text{ and } p = \frac{m^2 + n^2}{2mn}; \text{ whence } \frac{x}{y} = \frac{(m^2 - n^2)^2}{4m^2n^2}, \ \&c.$$

6. $x^2 - y^2 = a^2$: let $x + y = \frac{m}{n} a$, $x - y = \frac{n}{m} a$, whence $x$ and $y$.

7. $x + y = p^2$, $x - y = q^2$; $\therefore x = \frac{1}{2}(p^2 + q^2)$, $y = \frac{1}{2}(p^2 - q^2)$; for $p$ write $m + n$, for $q$ write $m - n$.

8. $x^2 + xy + y^2 = \square$, or $v^2 + v + 1 = \square = \left(v - \frac{m}{n}\right)^2$, if $v = \frac{x}{y}$; whence $v$, &c.

9. $x^2 + y^2 = a^2$, or $y^2 = a^2 - x^2$: put $y = \frac{m}{n}(a - x)$ and $y = \frac{n}{m}(a + x)$, whence $x$ and $y$.

10. $x^2 - ay^2 = z^2$, or $v^2 - a = \square = \left(v - \frac{m}{n}\right)^2$, if $v = \frac{x}{y}$; whence $x, y, z$.

11. Put $x + xy = n^2x^2$, $y + xy = m^2y^2$;
$\therefore y = n^2x - 1$, $x = m^2y - 1 = m^2n^2x - m^2 - 1$, whence $x$, &c.

12. $x^2 + y^2 = a^2 + b^2$, or $x^2 - a^2 = b^2 - y^2$;

$$\text{put } x + a = \frac{m}{n}(b - y),\ x - a = \frac{n}{m}(b + y).$$

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## Ex. 45.

2. Let $x$ = chance of one event, $\frac{2}{3}x$ of the other; $\therefore x + \frac{2}{3}x$ = certainty = 1.

7. (i) Leaving out the two marked ones, there will be 8 other balls, which may be taken, 2 together, in 28 ways; to each of these sets of 2 balls we may join the 2 marked ones, and thus we shall have 28 ways in which the 10 balls may be taken, 4 together, so that 2 of them may be the marked ones: but for 10 balls, $C_4 = 210$; $\therefore$ chance $= \frac{28}{210} = \frac{2}{15}$.

(ii) The 3 marked ones may be taken, 2 together, in 3 ways, and the 7 others may be taken, 2 together, in 21 ways; therefore the 10 may be taken, 4 together, so that 2 of them may be marked ones, in $3 \times 21 = 63$ ways; and, as before, $C_4$ for 10 balls $= 210$; $\therefore$ chance $= \frac{63}{210} = \frac{3}{10}$.

(iv) If 2 of the 6 marked ones are drawn, (for which $C_2 = 15$,) and 2 of the 4 others, (for which $C_2 = 6$,) we have the chance $= \frac{90}{210}$;

if 3 of the 6 marked ones ($C_3 = 20$) and 1 of the 4 others ($C_1 = 4$), the chance $= \frac{80}{210}$;

if 4 of the 6 marked ones ($C_4 = 15$), the chance $= \frac{15}{210}$;
$\therefore$ total chance $= \frac{185}{210} = \frac{37}{42}$.

8. (i) For the 10 leaders, $C_1 = 10$; for the 90 others, $C_1 = 90$; for the whole 100, $C_2 = 4950$; $\therefore$ chance $= \frac{900}{4950} = \frac{2}{11}$.

## Ex. 47.

1. Exp. $= \frac{1}{6}$ of $1d + \frac{1}{6}$ of $2d$ + &c. $= \frac{1}{6}$ of $21d = 3\frac{1}{2}d$; and the exp. being the same for *each* die, that for *two* dice will be $7d$.

2. Consider first the throws, omitting the doublets. Now 1 of the first die may fall with 2, 3, 4, 5, or 6 of the second, the chance of each of which being $\frac{1}{36}$, we have exp. from these throws $= \frac{1}{36}\{(1+2)+(1+3)+\text{\&c.}\} = \frac{1}{36}\{5 \times 1 + 20\}$; so exp., when 2 of the first die falls with 1, 3, 4, 5 or 6 of the second, $= \frac{1}{36}\{5 \times 2 + 19\}$, &c.; and whole exp. in pence (excluding the doublets, as aforesaid)

$$= \frac{1}{36}\{5(1+2+\text{\&c.}+6)+(20+19+\text{\&c.}+15)\} = 5\tfrac{5}{6}:$$

but exp. from the doublets in pence

$$= \frac{1}{36}\{4+16+36+64+100+144\} = 10\tfrac{1}{9};$$

whence $10\frac{1}{9}d + 5\frac{5}{6}d = 15\frac{17}{18}d$, $10\frac{1}{9}d - 5\frac{5}{6}d = 4\frac{5}{18}d$.

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## Ex. 48.

1. The cases possible are $W_4$ and $W_1B_3$, for which $P_1 = \frac{1}{2} = P_2$; $p_1 = 1$, $p_2 = \frac{1}{4}$, $Q_1 = \frac{4}{5}$, $Q_2 = \frac{1}{5}$: $\therefore$ chance that all are white $= Q_1 = \frac{4}{5}$, and chance that, the ball drawn having been replaced, $W$ will again be drawn, is $Q_1 = \frac{1}{4}Q_2 = \frac{17}{20}$.

2. As in (1), $Q_1$ (for $W_4$) $= \frac{4}{5}$, $Q_2$ (for $W_1B_3$) $= \frac{1}{5}$; $\therefore$ (i) chance $= \frac{3}{4}Q_2 = \frac{3}{20}$; (ii) the supposed occurrence is only possible with the case $W_1B_3$, and here, there being 6 ways in which the four can be drawn, two and two, in 3 of which ways $W$ may be drawn with one of the $B$'s, we have the chance $= \frac{3}{6} \times Q_2 = \frac{1}{10}$: (iii) chance $= \frac{3}{4} \times \frac{1}{3} \times Q_2 = \frac{1}{20}$.

3. The cases possible are $W_1B_2$, $W_2B_1$, $W_3$, (not $B_3$, because *one* ball at least, as we see by the Question, must be $W$):

$$P_1 = P_2 = P_3;\ p_1 = \tfrac{1}{9},\ p_2 = \tfrac{4}{9},\ p_3 = 1 = \tfrac{9}{9},\ \Sigma(p) = \tfrac{14}{9};$$

$$\therefore\ Q_1 = \tfrac{1}{14},\ Q_2 = \tfrac{4}{14},\ Q_3 = \tfrac{9}{14},\ \text{and chance} = \tfrac{4}{9}Q_1 + \tfrac{1}{9}Q_2 = \tfrac{4}{63}.$$

4. The cases possible are

$$W_2B_5,\ W_3B_4,\ W_4B_3,\ W_5B_2;\ \therefore\ P_1 = P_2 = \text{\&c.};$$

$$p_1 = \tfrac{1}{21},\ p_2 = \tfrac{3}{21},\ p_3 = \tfrac{6}{21},\ p_4 = \tfrac{10}{21},\ \Sigma(p) = \tfrac{20}{21};$$

$$\therefore\ Q_1 = \tfrac{1}{20},\ Q_2 = \tfrac{3}{20},\ Q_3 = \tfrac{6}{20},\ Q_4 = \tfrac{10}{20} = \tfrac{1}{2} = 1^{\text{st}}\ \text{Ans.}$$

If the two *white* ones drawn be removed, then $Q_1$, $Q_2$, $Q_3$, $Q_4$, are the chances that the remaining five are

$$B_5,\ W_1B_4,\ W_2B_3,\ W_3B_2,$$

respectively: estimating in each of these cases the chance that, if two balls are drawn they will be $BW$, we shall find the chance required $= \frac{2}{5}Q_2 + \frac{3}{5}Q_3 + \frac{3}{5}Q_4 = \frac{27}{50}$.

5. The cases possible are

$$W_2B_5,\ W_3B_4,\ W_4B_3,\ W_5B_2;\ P_1 = P_2 = \&\text{c.}:$$

(i) $p_1 = \frac{10}{21}$, $p_2 = \frac{6}{21}$, $p_3 = \frac{3}{21}$, $p_4 = \frac{1}{21}$, $\Sigma(p) = \frac{20}{21}$;

$$Q_1 = \tfrac{10}{20},\ Q_2 = \tfrac{6}{20},\ Q_3 = \tfrac{3}{20},\ Q_4 = \tfrac{1}{20} = 1^{\text{st}}\ \text{Ans.};$$

and, for the second chance, $Q_1$, $Q_2$, $Q_3$, $Q_4$ correspond now (when the two black ones are removed) to $W_2B_3$, $W_3B_2$, $W_4B_1$, $W_5$, respectively, and $\frac{3}{5}Q_1 + \frac{3}{5}Q_2 + \frac{2}{5}Q_3 = \frac{27}{50} = 2^{\text{nd}}$ Ans.

(ii) $p_1 = \frac{10}{21}$, $p_2 = \frac{12}{21}$, $p_3 = \frac{12}{21}$, $p_4 = \frac{10}{21}$, $\Sigma(p) = \frac{44}{21}$;

$$Q_1 = \tfrac{10}{44},\ Q_2 = \tfrac{12}{44},\ Q_3 = \tfrac{12}{44},\ Q_4 = \tfrac{10}{44} = \tfrac{5}{22} = 3^{\text{rd}}\ \text{Ans.};$$

and, for the second chance, $Q_1$, $Q_2$, $Q_3$, $Q_4$, correspond now (the white and black being removed) to $W_1B_4$, $W_2B_8$, $W_3B_2$, $W_4B_1$, respectively, and $\frac{4}{10}Q_1 + \frac{6}{10}Q_2 + \frac{6}{10}Q_3 + \frac{4}{10}Q_4 = \frac{28}{55} =$ 4th Ans.:

(iii) $p_1 = \frac{10}{42}$, $p_2 = \frac{12}{42}$, $p_3 = \frac{12}{42}$, $p_4 = \frac{10}{42}$, which being in the same ratio as in (ii), and the remaining circumstances being similar, the same answers will be obtained.

6. The cases possible will be $W_4$, $W_1B_3$, for which $P_1 = P_2$: $p_1 = 1$, $p_2 = \frac{1}{4}$, $\Sigma(p) = \frac{5}{4}$; $Q_1 = \frac{4}{5} = 1^{\text{st}}$ Ans., $Q_2 = \frac{1}{5}$, and $Q_1 + \frac{1}{4}Q_2 = \frac{17}{20}$.

7. The balls were originally either $W_3$ or $B_3$, for each of which the chance is $\frac{1}{2}$: hence when the ball is dropt in, which may be either $W$ or $B$, there will be four possible cases $W_4$, $W_3B_1$, $W_1B_3$, or $B_4$; and, since the chance of the fourth ball being $W$ or $B$ is $\frac{1}{4}$ or $\frac{3}{4}$, we have $P_1 = \frac{1}{8}$, $P_2 = \frac{3}{8}$, $P_3 = \frac{1}{8}$, $P_4 = \frac{3}{8}$, $p_1 = 1$, $p_2 = \frac{3}{4}$, $p_3 = \frac{1}{4}$, $p_4 = 0$, $P_1p_1 = \frac{1}{8}$, $P_2p_2 = \frac{9}{32}$, $P_3p_3 = \frac{1}{32}$, $P_4p_4 = 0$, $\Sigma(Pp) = \frac{14}{32}$; $\therefore\ Q_1 = \frac{2}{7} = 1^{\text{st}}$ Ans., $Q_2 = \frac{9}{14}$, $Q_3 = \frac{1}{14}$, $Q_4 = 0$, and $Q_1 + \frac{3}{4}Q_2 + \frac{1}{4}Q_3 = \frac{11}{14} = 2^{\text{nd}}$ Ans.

8. (i) $n - 1$ cases are *equally* possible,

$$W_1B_{n-1},\ W_2B_{n-2},\ \&\text{c.},\ W_{n-1}B_1;$$

$$p_1 = \frac{1}{n},\ p_2 = \frac{2}{n},\ \&\text{c.},\ p_{n-1} = \frac{n-1}{n};\ \ \therefore \Sigma(p) = \frac{n-1}{2};$$

$$\therefore Q_1 = \frac{2.1}{n(n-1)},\ Q_2 = \frac{2.2}{n(n-1)},\ \&\text{c.},\ Q_{n-1} = \frac{2(n-1)}{n(n-1)},$$

and chance required is

$$\frac{1}{n}Q_1 + \frac{2}{n}Q_2 + \&\text{c.} + \frac{n-1}{n}Q_{n-1} = \frac{2}{n(n-1)}\left\{\frac{1^2 + 2^2 + \&\text{c.} + (n-1)^2}{n}\right\} = \frac{2n-1}{3n}.$$

(ii) $n$ cases are *equally* possible, $W_1B_{n-1}$, $W_2B_{n-2}$, &c., $W_{n-1}B_1$, $W_n$, (not $B_n$, because *one* white ball is certainly in the bag, being drawn;) the rest of the reasoning as in (i).

9. The coin taken out is equally likely to be $S$ or $s$; hence the cases possible in $B$'s pocket are $S_2$, $s_2$, corresponding to which in $A$'s are $S_1s_3$, $S_3s_1$; $\therefore p_1 = \frac{3}{4}$, $p_2 = \frac{1}{4}$, $Q_1 = \frac{3}{4} =$ 1st Ans., and $B$'s exp. $= \frac{3}{4}$ of 40$s$ + $\frac{1}{4}$ of 2$s$ = £1 10$s$ 6$d$.

10. $A$'s purse contained at first $S_2s_1$, $B$'s contained $S_1s_2$: now the coins transferred from $A$'s and $B$'s may be $S$ and $S$, or $S$ and $s$, or $s$ and $S$, or $s$ and $s$, for which $P_1 = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$, $P_2 = \frac{4}{9}$, $P_3 = \frac{1}{9}$, $P_4 = \frac{2}{9}$, and, corresponding to these suppositions, $A$'s will now contain $S_2s_1$, $S_1s_2$, $S_3$, $S_2s_1$, respectively, and $B$'s $S_1s_2$, $S_2s_1$, $s_3$, $S_1s_2$; therefore

(i) $A$'s exp. $= \frac{2}{9}$ of 41$s$ + $\frac{4}{9}$ of 22$s$ + $\frac{1}{9}$ of 60$s$ + $\frac{2}{9}$ of 41$s$ = £1 14$s$ 8$d$;
$B$'s exp. = remr of the whole £3 3$s$ = £1 8$s$ 4$d$:

(ii) $p_1 = \frac{2}{3}, p_2 = \frac{1}{3}, p_3 = 1, p_4 = \frac{2}{3}$; $P_1p_1 = \frac{4}{27}$, $P_2p_2 = \frac{4}{27}$, $P_3p_3 = \frac{3}{27}$, $P_4p_4 = \frac{4}{27}$, $\therefore Q_1 = \frac{4}{15}$, &c., and $A$'s exp. $= \frac{4}{15}$ of 41$s$ + &c.:

(iii) $p_1 = \frac{1}{3}$, $p_2 = \frac{2}{3}$, $p_3 = 0$, $p_4 = \frac{1}{3}$; $\therefore P_1p_1 = \frac{2}{27}$, &c.

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## Ex. 49.

1. (i) $(x + x^2 + \&c. + x^6)^2 = x^2\left(\dfrac{1-x^6}{1-x}\right)^2 = x^2(1-2x^6)(1+2x+3x^2+\&c.)$, omitting $x^{12}$ in the numr, which cannot affect the chances of the question; hence coeff. of $x^5 = 4$, of $x^7 = 6$, of $x^9 = 8 - 4 = 4$, of $x^{10} = 9 - 6 = 3$, and the chances are $\frac{4}{36}$, &c.

(ii) $x^3\left(\dfrac{1-x^6}{1-x}\right)^3 = x^3(1 - 3x^6 + 3x^{12} - \&c.) \times$
$(1+3x+6x^2+10x^3+15x^4+21x^5+28x^6+36x^7+45x^8+\&c+105x^{13}+\&c)$;
hence coeff. of $x^5 = 6$, of $x^7 = 15$, of $x^9 = 28 - 3 = 25$, &c.

3. In the above (ii), the sum of coeffs of $x^3$, $x^4$, &c., $x^{10}$, is
$$1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 - 3 - 9 = 108;$$
hence chance of throwing 10 or under $= \frac{108}{216} = \frac{1}{2}$.

4. As in the preceding, chance of throwing 9 or under is $\frac{126}{1296}$; $\therefore$ chance of throwing above 9 is $\frac{1170}{1296}$.

5. The fact *has* happened, or it has *not;* $P_1 = P_2$; let $a = \frac{2}{3}$, $b = \frac{3}{4}$, be the *veracities* of $A$ and $B$; $a' = 1 - a = \frac{1}{3}$, $b' = \frac{1}{4}$; then
(i) $p_1 = ab = \frac{1}{2}$, $p_2 = a'b' = \frac{1}{12}$, $Q_1 = \frac{6}{7}$;
(ii) $p_1 = ab' = \frac{1}{6}$, $p_2 = a'b = \frac{1}{4}$, $Q_1 = \frac{2}{5}$;
(iii) $p_1 = ab + a'b' = \frac{7}{12}$, $p_3 = ab' + a'b = \frac{5}{12}$; $Q_1 = \frac{7}{12}$.

6. In (i) $p_1$ = prob. of all speaking truth $= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$, $p_2$ = prob. of all speaking untruth $= \frac{8}{125}$; $\therefore Q_1 = \frac{27}{35}$: &c.

7. In (i) the chance of its being true is $p^3 + 3pq^2 = \frac{63}{125}$; in (ii) it is $abc + ab'c' + a'bc' + a'b'c = \frac{8}{35} + \frac{4}{35} + \frac{3}{35} + \frac{8}{105} = \frac{53}{105}$.

8. $ab' + a'b = \frac{7}{15}$.

9. (i) $p_1 = ab = \frac{6}{15}$, $p_2 = a'b' = \frac{2}{15}$, $Q_1 = \frac{3}{4}$, and exp. $= \frac{3}{4}$ of £10 = £$7\frac{1}{2}$:
(ii) $p_1 = ab' = \frac{4}{15}$, $p_2 = a'b = \frac{3}{15}$, $Q_1 = \frac{4}{7}$, &c.:
(iii) $p_1 = ab + a'b' = \frac{8}{15}$, $p_2 = \frac{7}{15}$, &c.

10. The reasoning being the same in each case, take that of (iii): here prob. of truth of $A$'s statement is $ac + a'c' = \frac{9}{20}$, and of $B$'s, is $bc + b'c' = \frac{7}{12}$: these we may now treat as the actual *veracities* of $A$ and $B$, so that

$$p_1 = \frac{9}{20} \times \frac{7}{12} = \frac{63}{240}, \quad p_2 = \frac{11}{20} \times \frac{5}{12} = \frac{55}{240}, \quad Q_1 = \frac{63}{118}.$$

11. (i) $p_1 = abc = \frac{1}{5}$, $p_2 = a'b'c' = \frac{1}{20}$, $Q_1 = \frac{4}{5}$;
(ii) $p_1 = ab'c' = \frac{1}{30}$, $p_2 = a'bc = \frac{3}{10}$, $Q_1 = \frac{1}{10}$.

12. (i) The chance is $abc + ab'c' + a'bc' + a'b'c = \frac{29}{60}$;
(ii) the prob. that $B$ and $C$ will speak *truth* is $bc = \frac{6}{12}$, or *untruth*, is $b'c' = \frac{1}{12}$; $\therefore$ chance of $A$'s getting to know the truth is $\frac{6}{7}$; and the chance required is $\frac{6}{7}a + \frac{1}{7}a' = \frac{3}{7}$.

13. It was either *white* or *red*: hence in (iii), (the reasoning being alike in all the cases,) $P_1 = \frac{2}{7}$, $P_2 = \frac{5}{7}$, $p_1 = abc' = \frac{1}{15}$, $p_2 = a'b'c = \frac{3}{20}$, $P_1p_1 = \frac{2}{105}$, $P_2p_2 = \frac{3}{28}$; $\therefore Q_1 = \frac{8}{53}$.

14. (i) $p_1 = p^4$ (where $p = \frac{2}{3}$, $q = \frac{1}{3}$,) $= \frac{16}{81}$, $p_2 = q^4 = \frac{1}{81}$; $Q_1 = \frac{16}{17}$;
(ii) the chance is $p^4 + 6p^2q^2 + q^4 = \frac{41}{81}$; (iii) prob. of truth of $A$'s assertion $= p^3 + 3pq^2 = \frac{14}{27}$ = d° of $B$'s, which may now be treated as their actual *veracities*, $a$ and $b$; $\therefore p_1 = ab = \frac{196}{729}$, $p_2 = a'b' = \frac{169}{729}$, $Q_1 = \frac{196}{365}$;
(iv) $p_1 = ab' = \frac{182}{729}$, $p_2 = a'b = \frac{182}{729}$, $Q_1 = \frac{1}{2}$.

15. Here $p = \frac{4}{5}$, $q = \frac{1}{5}$: (i) $p_1 = p^3q = \frac{64}{625}$, $p_2 = pq^3 = \frac{4}{625}$, $Q_1 = \frac{16}{17}$:
(ii) prob. of truth of $A$'s assertion, who has received by tradition from $D$, is $p^2 + q^2 = \frac{17}{25}$ = d° of $B$'s = d° of $C$'s; $\therefore p_1 = \frac{17}{25} \times \frac{17}{25} \times \frac{8}{25}$, $p_2 = \frac{8}{25} \times \frac{8}{25} \times \frac{17}{25}$, $Q_1 = \frac{17}{25}$.

16. (i) $p_1 = abc' = \frac{1}{10}$, $p_2 = a'b'c = \frac{1}{10}$, $Q_1 = \frac{1}{2}$;
(ii) $p_1 = ab'c' = \frac{1}{15}$, $p_2 = a'bc = \frac{3}{20}$, $Q_1 = \frac{4}{13}$;
(iii) $p_1 = abc + ab'c' + a'bc' + a'b'c = \frac{3}{10} + \frac{1}{15} + \frac{1}{20} + \frac{1}{10} = \frac{31}{60}$;
(iv) prob. of truth of $A$'s assertion $= ac + a'c' = \frac{7}{12}$, d° of $B$'s $= bc + b'c' = \frac{11}{20}$, which numbers we may now take as the measures of their veracities; $\therefore p_1 = \frac{7}{12} \times \frac{9}{20}$, $p_2 = \frac{5}{12} \times \frac{11}{20}$, $Q_1 = \frac{63}{118}$.

## MISCELLANEOUS EXAMPLES.—PART II.

3. $(a-b)^2=(a-b)\{(a-c)-(b-c)\}=(a-b)(a-c)+(b-a)(b-c)$, &c.

6. $a+\beta=-p,\ a\beta=q:\ \therefore\ a^2+a\beta+\beta^2=(a+\beta)^2-a\beta=p^2-q$; $a^3+\beta^3=(a+\beta)^3-3a\beta(a+\beta)=-p^3+3pq$; $a^4+a^2\beta^2+\beta^4$ $=(a^2+a\beta+\beta^2)(a^2-a\beta+\beta^2)=\{(a+\beta)^2-a\beta\}\{(a+\beta)^2-3a\beta\}$ $=(p^2-q)(p^2-3q)$.

7. Let $x+2x+3x+\text{\&c.}=x(1+2+3+\text{\&c.})=x\,.\,\dfrac{n(n+1)}{2}=a$.

8. $(m+n)(m+n-1)=56$, or $(m+n)^2-(m+n)-56=0$, whence $m+n=8$; $(m-n)(m-n-1)=12$, or $(m-n)^2-(m-n)-12=0$, whence $m-n=4$.

10. $x+y+z=P,\ xR^a=yR^b=zR^c;\ \therefore\ y=x\dfrac{R^a}{R^b},\ z=x\dfrac{R^a}{R^c}$, and $x\left(1+\dfrac{R^a}{R^b}+\dfrac{R^a}{R^c}\right)=P$.

12. Observe that $m^2-1=\frac{1}{4}(x^2+2+x^{-2})-1=\frac{1}{4}(x^2-2+x^{-2})=\frac{1}{4}(x-x^{-1})^2$, &c.; $\therefore\ mn+\sqrt{\{(m^2-1)(n^2-1)\}}=\frac{1}{4}(x+x^{-1})(y+y^{-1})+\frac{1}{4}(x-x^{-1})(y-y^{-1})=\text{\&c.}$

13. Put $n=0$, which gives the *first* term $=1\frac{1}{3}$; put $n=1$, which gives the sum of *first* and *second* terms $=4\frac{2}{3}$; $\therefore$ common difference $=2$, &c.

14. (1) $x+3y=6$, (2) $(x+\frac{1}{10}x)+(3y+\frac{3}{2}y)=7$.

15. $\dfrac{a_1}{a_n}=\dfrac{a_1}{a_2}\times\dfrac{a_2}{a_3}\times\dfrac{a_3}{a_4}\times\text{\&c.}\times\dfrac{a_{n-1}}{a_n}=\dfrac{a_1}{a_2}\times\dfrac{a_1}{a_2}\times\dfrac{a_1}{a_2}\times\text{\&c.}\times\dfrac{a_1}{a_2}=\left(\dfrac{a_1}{a_2}\right)^{n-1}$.

16. The L. C. M. of 2, 3, 4, 5, 6, being 60, we must have $N=60x$ $=$ also $7y+5$; $\therefore\ 60x-7y=5$, an ind. equation, whence we get $x=3,\ N=180$.

17. Take $x$ gals from $A$, $y$ gals from $B$: then $\frac{1}{4}x$ is wine, $\frac{3}{4}x$ is water; so $\frac{3}{8}y$ is wine, $\frac{5}{8}y$ is water;

$$\therefore\ (1)\ \tfrac{1}{4}x+\tfrac{3}{8}y=5,\quad (2)\ \tfrac{3}{4}x+\tfrac{5}{8}y=9.$$

18. The quantity does not satisfy the *criterion* of [33]; but

$$\sqrt{\left\{1+\frac{1}{\sqrt{(1-c^2)}}\right\}}=\sqrt{\frac{1+\sqrt{(1-c^2)}}{\sqrt{(1-c^2)}}}=\frac{1}{\sqrt[4]{(1-c^2)}}\sqrt{\{1+\sqrt{(1-c^2)}\}}$$

$$=\frac{1}{\sqrt[4]{(1-c^2)}}\left\{\sqrt{\tfrac{1}{2}(1+c)}+\sqrt{\tfrac{1}{2}(1-c)}\right\}\text{ by (119)}$$

$$=\sqrt[4]{\frac{(1+c)^2}{4(1-c^2)}}+\sqrt[4]{\frac{(1-c)^2}{4(1-c^2)}}=\text{\&c.}$$

19. $\frac{x^2}{a+x} + a\frac{x}{\sqrt{(a+x)}} = b$, or $z^2 + az - b = 0$, if $z = \frac{x}{\sqrt{(a+x)}}$;
$\therefore z = \frac{1}{2}\{-a \pm \sqrt{(a^2 + 4b^2)}\} = p$ suppose, or $\frac{x}{\sqrt{(a+x)}} = p$, whence
$x^2 - px - ap = 0$.

20. *A* the spectator's place; *AB*, the hor. distance to foot of tower, $= x$; $BC = a$, $CD = c$; then

$$\tan BAD = \tan\theta = \frac{a+c}{x}, \quad \tan BAC = \tan\phi = \frac{a}{x}$$

and $\therefore \tan CAB = \tan(\theta - \phi) = \frac{cx}{x^2 + a(a+c)} = m$, suppose;

$\therefore mx^2 - cx + ma(a+c) = 0$, and $x = \frac{c \pm \sqrt{\{c^2 - 4m^2a(a+c)\}}}{2m}$,

where [77] the greatest admissible value of $m$ is when

$$c^2 = 4m^2a(a+c), \text{ or } 2m = \frac{c}{\sqrt{\{a(a+c)\}}},$$

in which case $x = \frac{c}{2m} = \sqrt{\{a(a+c)\}}$.

21. $$\text{It} = \{1 + \frac{a^2 + b^2 - c^2 - d^2}{2(ab+cd)}\} \times \{1 - \frac{a^2 + b^2 - c^2 - d^2}{2(ab+cd)}\}$$
$$= \frac{(a+b)^2 - (c-d)^2}{2(ab+cd)} \times \frac{(c+d)^2 - (a-b)^2}{2(ab+cd)} = \&\text{c.}$$

23. Prove that $\frac{a^2 - b^2}{a^2} = \frac{c^2 - d^2}{c^2}$; then $\frac{a+b}{c+d} = \frac{a^2(c-d)}{c^2(a-b)}$.

24. $S_n = n^2 =$ sum of $n$ terms $=$ sum of *first* half of $2n$ terms, $S_{2n} = (2n)^2 = 4n^2$; $\therefore S_{2n} - S_n = 3n^2 =$ sum of *second* half of $2n$ terms, which is to the former sum as 3 : 1.

25. Let $x =$ distance, $y =$ rate; therefore

$$(1)\ \frac{x}{y + \frac{1}{2}} = \frac{4}{5}\left(\frac{x}{y}\right), \quad (2)\ \frac{x}{y - \frac{1}{2}} = \frac{x}{y} + 2\tfrac{1}{2}.$$

26. The five *different* letters, $n, o, t, a, i$, taken 3 together, give, as usual, 10 Combns: besides these, there are the Combns of the form *nno*, in which two *like* letters occur, and of these 4 can be formed with each of the three pairs of double letters, since each pair may be taken with one of the four single letters: these will be, therefore, in all $10 + 3 \times 4 = 22$ Combns.

27. Let $x =$ n° of £'s, $y$ of shillings, $z$ of sixpences: then (in sixpences) $40x + 2y + z = 333$, and $2y = 40x - 42 = z + 63$.

28. Since the *rate* is mentioned, use the formula in [199] for the *true* time.

29. $\frac{(1)}{(2)^2}$ gives $\frac{(x^2-xy+y^2)(x-y)^2}{(x+y)^4}=\frac{19}{1024}=\frac{1-3v+4v^2-3v^3+v^4}{1+4v+6v^2+4v^3+v^4}$, if $v=\frac{y}{x}$; $\therefore\ 1005(v^4+1)-3148(v+v^3)+3982v^2=0$, a recurring equation, whence (as usual) we get by [61]

$$v+v^{-1}=\tfrac{34}{15},\text{ and } v=\tfrac{3}{5}\text{: put } y=\tfrac{3}{5}x \text{ in (2).}$$

30. Let $P$ = population at beginning of *first* year;

$\therefore\ P\left(1+\frac{1}{m}-\frac{1}{n}\right)$ = d° at beginning of *second* year = $P'$, suppose;

$P'\left(1+\frac{1}{m}-\frac{1}{n}\right)=P\left(1+\frac{1}{m}-\frac{1}{n}\right)^2$ = d° for *third* year = $P''$, &c.,

and, generally, $P\left(1+\frac{1}{m}-\frac{1}{n}\right)^x$ = d° at end of $x^{\text{th}}$ year = $2P$, by Question.

31. In (i) $x^2=a\pm\sqrt{(a^2-b^2)}=\frac{1}{2}[2a\pm2\sqrt{\{(a+b)(a-b)\}}]$;

$$\therefore\ x=\frac{1}{\sqrt{2}}\{\sqrt{(a+b)}\pm\sqrt{(a-b)}\}:$$

In (ii) $x^2=\frac{1}{2}a^2\{(n^2+1)\pm\sqrt{(n^4+n^2+1)}\}$

$$=\tfrac{1}{4}a^2[2(n^2+1)\pm2\sqrt{\{(n^2+n+1)(n^2-n+1)\}}];$$

$$\therefore\ x=\pm\tfrac{1}{2}a\{\sqrt{(n^2+n+1)}\pm\sqrt{(n^2-n+1)}\}.$$

32. (i) Put $px^3+qx^2+rx+s=(p^{\frac{1}{3}}x+s^{\frac{1}{3}})^3$; expand, and equate coefficients.

(ii) Put $x^4+\&c.=(x^2+\frac{1}{2}px\pm s^{\frac{1}{4}})^2$, this being (plainly, from the second and last terms,) the form of the root to be assumed.

33. Let $x$ = length of line, $r$ = rate *per* hour at first;
then $x-r$ = distance remaining at end of first hour;

$$\therefore (1)\quad 1+1+\frac{x-r}{\frac{3}{5}r}=\frac{x}{r}+3,$$

$$\text{and (2)}\quad \left(1+\frac{50}{r}\right)+1+\frac{x-(r+50)}{\frac{3}{5}r}=\frac{x}{r}+1\tfrac{2}{3}.$$

34. $S=x(1+2+3+\&c. \text{ to } n \text{ terms})$

$$-\frac{1}{n-1}\{1.2+2.3+3.4+\&c. \text{ to } (n-1) \text{ terms}\}$$

$$=x\,.\,n\frac{n+1}{2}-\frac{1}{n-1}\cdot\frac{n(n^2-1)}{3} \text{ as in [184 Ex. 3]:}$$

putting this value of $S=0$, we get $x=\frac{2}{3}$.

35. $N=7x+1=8y+2=9z+3$, and $x+y+z=570$: find $x, y, z$, and then $N$. Or thus: $\frac{1}{7}N=x+\frac{1}{7}$, $\frac{1}{8}N=y+\frac{1}{4}$, $\frac{1}{9}N=z+\frac{1}{3}$; $\therefore N(\frac{1}{7}+\frac{1}{8}+\frac{1}{9})=570+\frac{1}{7}+\frac{1}{4}+\frac{1}{3}$, whence $N$ at once.

36. $\log_e \frac{1}{1-ax+x^2}=-\log_e\{1-x(a-x)\}=x(a-x)+\frac{x^2(a-x)^2}{2}+\&c.$ $+\frac{x^{n-2}(a-x)^{n-2}}{n-2}+\frac{x^{n-1}(a-x)^{n-1}}{n-1}+\frac{x^n(a-x)^n}{n}+\&c.$, by [119]

where coeff. of $x^n=\frac{a^n}{n}-\frac{(n-1)\,a^{n-2}}{n-1}+\frac{\frac{(n-2)(n-3)}{1.2}\,a^{n-4}}{n-2}-\&c.$

38. From $\log\frac{5}{4}=\log\frac{10}{8}=1-3\log 2$, we get $\log 2$;
from $\log\frac{1}{9}=-2\log 3$, we get $\log 3$;
and by means of these we can easily find the first five of the required logs: the last $=\log 3-\log 40+\frac{3}{5}\log .0027-\frac{5}{3}\log .015$ $=.4771213+\bar{2}.3979400+\bar{2}.4588183+3.0398479$.

39. $x+y=2\{\sqrt{(ax)}+\sqrt{(by)}\}$, whence $(\sqrt{x}-\sqrt{a})^2+(\sqrt{y}-\sqrt{b})^2=a+b$: but $\sqrt{(ax)}+\sqrt{(by)}=a+b$, or $\sqrt{b}\,(\sqrt{y}-\sqrt{b})=\sqrt{a}\,(\sqrt{a}-\sqrt{x})$; substitute in the former the value from the last for $\sqrt{y}-\sqrt{b}$.

40. See [219 Ex. 1].

41. (i) $N(N^4-1)=(N-1)N(N+1)(N^2+1)$, where one of the first three factors, being consecutive numbers, is div. by 3; also $N$, being odd, must be of the form $2m+1$; $\therefore$ the product takes the form $2m(2m+1)(2m+2)(4m\ +4m+2)$, where either $2m$ or $2m+2=2(m+1)$ is div. by 4, and therefore the whole by $3\times 8\times 2=48$.

(ii) $N(N^2+20)=2m(4m^2+20)=8m(m^2+5)$, where $m$ must be of the form $6n$, or $6n\pm 1$, or $6n\pm 2$, or $6n+3$, by putting each of which for $m$ we prove the statement.

42. $\frac{1}{2x}\left\{1+\frac{4x}{x-1}\right\}^{\frac{1}{2}}=\frac{1}{2x}=\frac{1}{x-1}+$ (terms with $x$, $x^2$, &c. in num$^r$.) $=-1$, when $x=0$.

43. $9x+15(n-x)=12p$.

44. The quantity is $a^{-\frac{2}{xy}}+b^{\frac{1}{z}}a^{\frac{1}{yz}}+2b^{\frac{1}{2z}}a^{\frac{x-2z}{2xyz}}$.

45. $(5x+4)+(6y+4)=75$, or $5x+6y=67$, an Ind. Equation.

47. Put $\frac{a}{b}=x=\frac{c}{d}$; then $a=bx$, $c=dx$, and $\frac{a^2+b^2}{c^2+d^2}=\frac{b^2(x^2+1)}{d^2(x^2+1)}=\frac{b^2}{d^2}$,
$=$ (i) $\frac{b^2x}{d^2x}=\frac{ab}{cd}$, or $=$ (ii) $\frac{b^2(x+1)^2}{d^2(x+1)^2}=\frac{(bx+b)^2}{(dx+d)^2}=\frac{(a+b)^2}{(c+d)^2}$, or
$=$ (iii) $\frac{b^2(mx^2-nx+p)}{d^2(mx^2-nx+p)}=\&c.$

48. We may here use the formula for Comp. Int., $M = PR^n$; $\therefore 3P = PR^{100}$, or $\log R = \frac{1}{100} \log 3 = .0047712$, where the *mantissa* lies between the two given ones for 101.1 and 101.11, or for 10110 and 10111, and $R$ is found, as in [114], to be $1.011046 = 1 + r$; $\therefore r = .011046$ = annual increase for *one*, and annual increase for a *million* = 11046.

49. (i) $\left(\dfrac{a+x}{a-x}\right)^2 - 1 = \dfrac{4ax}{(a-x)^2} = \dfrac{cx}{ab}$; $\therefore x = 0$, and $\pm \dfrac{2\sqrt{a}}{a-x} = \sqrt{\dfrac{c}{ab}}$,

(ii) $a^3x^3 = b^3y^3 = c^3z^3$; $\therefore y = \dfrac{a}{b}x$, $z = \dfrac{a}{c}x$, which put in $\dfrac{a^3x}{y^2z^2} = 1$.

51. Put the equation in the form $P^2 - Q^2 = (P+Q)(P-Q) = 0$; then $P + Q = (a^2+bc)(b^2+ac)(c^2+ab) + (a^2-bc)(b^2-ac)(c^2-ab)$ $= (c^2+ab)\{a^2b^2 + c(a^3+b^3) + abc^2\} + (c^2-ab)\{a^2b^2 - c(a^3+b^3) + abc^2\}$ $= 2c^2(a^2b^2 + abc^2) + 2abc(a^3+b^3) = 2abc(a^3+b^3+c^3+abc)$: so $P - Q = 2c^3(a^3+b^3) + 2ab(a^2b^2+abc^2) = 2a^3b^3c^3(a^{-3}+b^{-3}+c^{-3}+a^{-1}b^{-1}c^{-1})$.

52. Observe that the numbers must be of the form $3n+1$ and $3n+2$; in (ii), $(N^3 \sim N'^3) - 7 = 27n(n+1)$, where either $n$ or $n+1$ must be even.

53. In (ii) $\dfrac{u}{v} = \dfrac{(x-1)(xy+x+1)}{(x-1)(xy-x+1)}$ = &c.

55. By [95 Cor.], the n° of terms $= \dfrac{3.4.5 \ldots 12.13.14}{1.2.3 \ldots 10.11.12} = \dfrac{13.14}{1.2} = 91$:

by [98], coeff. of $a^8(b^2)^2(c^3)^2 = \dfrac{\underline{|12}}{\underline{|8}\,\underline{|2}\,\underline{|2}} = \dfrac{9.10.11.12}{4} = 2970$.

56. Put $(1+x)^{\frac{3}{2}} = 1 + ax + bx^2 + cx^3 + dx^4 +$ &c.; square both sides, and equate coefficients.

57. Put $n = 1$, which gives the *first* term $= p + q$; again put $n = 2$, which gives the sum of *first* and *second* terms $= 2p + 4q$, whence $d = 2q$, &c.

58. $V_1$ = present value of £$a$ = by (198) $aR^{-m}$, $V_2 = bR^{-n}$; therefore, if $B$ return $A$ a bill of £$x$, its present value, or $xR^{-p}$, $= aR^{-m} - bR^{-n}$.

59. $x^2 \cdot \dfrac{2(x^2+1)}{(x^2-1)^2} = n(n-1)$, whence $(n^2-n-2)x^4 - 2(n^2-n+1)x^2 + n^2 - n = 0$; $\therefore x^2 = \dfrac{(n^2-n+1) \pm (2n-1)}{n^2-n-2} = \dfrac{n}{n-2}$ or $\dfrac{n-1}{n+1}$.

60. See [216 Ex. 3].

61. The G. C. M. of two quantities will (63) also divide their sum and difference: now

$$P+Q=2(a^2x^2+b^2y^2)-(x+z)(2a^2x-2b^2y)+2(a^2+b^2)xz=2b^2(y+x)(y+z),$$
$$P-Q=4abxy-(x+z)(2aby-2abx)-4abxz \qquad =2ab(x+y)(x-z);$$

these have the common factor $2b(x+y)$; $\therefore$ $2P$ and $2Q$ have the same, and therefore $P$ and $Q$ (which have evidently *not* a common factor 2) will have the G. C. M. $b(x+y)$.

62. (ii) $1-e^{\frac{a-x}{a}} = \dfrac{1-\{1+\frac{a-x}{a}+\frac{(a-x)^2}{1.2a^2}+\&c.\}}{(x-a)(x^3+ax^2+a^2x+a^3)} = \dfrac{\frac{1}{a}+\frac{a-x}{1.2a^2}+\&c.}{x^3+ax^2+a^2x+a^3}$

$=\dfrac{1}{4a^4}$, if $x=a$.

63. See [77].

64. $30x+10y+200z=2000$, or $3x+y+20z=200$, and $x+y+z=100$: eliminate $y$; then $2x+19z=100$, whence $z=2t$, $x=50-19t$, and $x+z=50-17t$, where (by Question) we must take $t=2$, since $x+z<20$.

65. $\dfrac{ma+nb}{m+n}=\sqrt{(ab)}=\frac{1}{2}(m+n)$; $\therefore$ $m+n=2\sqrt{(ab)}$, and $ma+nb=2ab$, between which equations eliminating $n$, we have

$$m=\frac{2b\{a-\sqrt{(ab)}\}}{a-b}=\frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}.$$

66. Reasoning as in [98 Cor.], we have here $\alpha+\beta+\gamma+\delta=7$, and

in (i) $\frac{1}{2}\beta+\gamma+\frac{1}{4}\delta=\frac{3}{2}$, or $2\beta+4\gamma+\delta=6$,
in (ii) $\frac{1}{2}\beta+\gamma+\frac{1}{4}\delta=2$, or $2\beta+4\gamma+\delta=8$.

67. Let $x$ = whole distance from $A$ to $B$, $y$ = rate per hour; then the *first* meeting is at $q+n$ hours after *midnight*, when the mail from $B$ has gone $ny$ miles, and that from $A$, $x-ny$ miles, for which it will have taken $\dfrac{x}{y}-n$ hours, and it is, consequently, now $\dfrac{x}{y}-n-(12-p)$ hours after *midnight;*

$$\therefore (1)\quad \frac{x}{y}-n-(12-p)=q+n.$$

The *second* meeting is when a second mail from $A$, (which, of course, left $A$ 24 hours after the former,) has gone $a$ miles, (that is, it will be at $p+\dfrac{a}{y}$ hours after *midday*,) and also when the mail from $B$ has gone $x-a$ miles, (that is, at $\frac{x-a}{y}-(12-q)$ hours after *midday;*) $\therefore$ (2) $p+\frac{a}{y}=\frac{x-a}{y}-(12-q)$: from (1) $\frac{x}{y}=q+2n+12-p$, from (2) $\frac{x-2a}{y}=p+12-q$;

$$\therefore \frac{x}{x-2a}=\frac{q+2n+12-p}{p+12-q}, \text{ and } \frac{x}{2a}=\frac{q+2n+12-p}{2q+2n-2p}.$$

68. $x^2=ay\{1-\frac{y}{a}+\frac{y^2}{a^2}-\&c.\}=ay\div(1+\frac{y}{a})=\frac{a^2y}{a+y}$;

$\therefore \frac{y}{a+y}=\frac{x^2}{a^2}$, and $\frac{y}{a}=\frac{x^2}{a^2-x^2}=\frac{x^2}{a^2}\{1+\frac{x^2}{a^2}+\frac{x^4}{a^4}+\&c.\}=\&c.$

69. Squaring, we get $2(x^2+a^2)-2\sqrt{\{(x^2+a^2)^2-4b^2x^2\}}=4(x^2-b^2)$, or $a^2+2b^2-x^2=\sqrt{\{x^4+4(a^2-b^2)x^2+a^4\}}$: square again.

71. The sum becomes $\frac{2(x^3+1)}{(x^2-x+1)(x^2+x+1)}=\frac{2(x+1)}{x^2+x+1}$.

72. Multiply each fraction, numerator and denominator, by $\sqrt{2}$; and then observe that, in the denominators, $\sqrt{(4\pm2\sqrt{3})}=\sqrt{3}\pm1$.

73. Since $c^2=ab$, put $\frac{a}{c}=\frac{c}{b}=x$; then $\frac{a}{b}=\frac{a}{c}\times\frac{c}{b}=x^2=$

(i) $\frac{b^2x^2+bcx^2+c^2x^2}{b^2+bc+c^2}=\frac{c^2+ac+a^2}{b^2+bc+c^2}$, or = (ii) $\frac{(bx+cx)^2}{(b+c)^2}=\frac{(c+a)^2}{(b+c)^2}$.

75. $\frac{(x+a)(x-a)(x-2a)}{(x+2a)(x-a)(x-2a)}$ becomes a vanishing fraction, if $x=a$ or if $x=2a$; substitute these values in $\frac{x+a}{x+2a}$.

76. $S=\frac{a}{1-r}$, $s^2=a^2+a^2r^2+a^2r^4+\&c=\frac{a^2}{1-r^2}$; $\therefore \frac{S^2}{s^2}=\frac{1-r^2}{(1-r)^2}=\frac{1+r}{1-r}$;

hence $r=\frac{S^2-s^2}{S^2+s^2}$, and the sum required $=a\frac{1-r^n}{1-r}=S(1-r^n)=\&c.$

77. In an *unit* of time, $A$ and $A'$ will do respectively $\frac{a}{t}$ and $\frac{a'}{t'}$ of work, or if they do together the work $c$ in time $x$, then $\frac{c}{x}$ = work done by them together in an unit of time:

$\therefore \frac{a}{t}+\frac{a'}{t'}=\frac{c}{x}$, whence $x=\frac{ctt'}{at'+a't}$. If $c=a+a'$, then

$x=\frac{(a+a')tt'}{at'+a't}$, = (i) $\frac{(t+t')tt'}{tt'+t't}$ or $\frac{t+t'}{2}$, = (ii) $\frac{(\sqrt{t}+\sqrt{t'})tt'}{t'\sqrt{t}+t\sqrt{t'}}$ or $\sqrt{(tt')}$, which results are obtained by putting for $a$ and $a$ their *proportional* values by [27].

78. The n° of Combinations required is plainly the same as the n° of terms in the product

$$(1+a+a^2+\&c.+a^p)\,(1+b+b^2+\&c.+b^p)\,(1+c+c^2+\&c.+c^r)\ldots,$$

with the exception of *one* term, viz. the term 1, which contains no power of $a$, $b$, $c$, &c: hence the number of Comb$^{ns}$ required is $(p+1)\,(q+1)\,(r+1)\ldots-1$.

79. From (2) $y=\dfrac{a^3}{a^2-bx}$: substitute for $y$ in $ax+\dfrac{b^3}{y}=a^2$:

$$\text{then } x=\frac{a^2\,(a^2+ab+b^2)}{a^3+a^2b+ab^2+b^3}=\frac{a^2\,(a^3-b^3)}{a^4-b^4}.$$

80. See [217 Ex. 2].

81. Expand the quantities, and apply [Ex. **15** 8, 9].

82. $c'(ax+by)=c\,(a'x+b'y)$, whence $\dfrac{x}{y}=\dfrac{b'c-c'b}{ac'-a'c}$: but we have $c^2\,(x^2+y^2)=(ax+by)^2$; $\therefore$ by [27], writing for $x$ and $y$ their *proportional* quantities in this *homogeneous* equation, we get

$$c^2\{(b'c-c'b)^2+(ac'-a'c)^2\}=\{a(b'c-c'b)+b(ac'-a'c)\}^2=c^2(ab'-a'b)^2.$$

85. (ii) $b=\dfrac{2ac}{a+c}$; $\therefore b^2=\dfrac{4a^2c^2}{(a+c)^2}=\dfrac{4a^2c^2}{a^2+c^2+2ac}<\dfrac{4a^2c^2}{4ac}<ac<\dfrac{a^2+c^2}{2}$.

86. $AB=AC=x=$ side of given square: bisect $AB$, $AC$, in $D$, $E$; $DE=$ side of second square $=x-1$: then $DE^2=AD^2+AE^2=\frac{1}{2}AB^2$, or $DE=\dfrac{AB}{\sqrt{2}}$, whence $x-1=\dfrac{x}{\sqrt{2}}$, or $x=2+\sqrt{2}$; so side of third square $=\dfrac{DE}{\sqrt{2}}$, &c: hence sum of perimeters $=4\,(AB+DE+\&c)=4AB\,(1+\dfrac{1}{\sqrt{2}}+\&c)$ $=4AB(2+\sqrt{2})$ by $(146)=4\,(2+\sqrt{2})^2=46.6$ ft nearly; sum of areas $=AB^2+DE^2+\&c.=AB^2(1+\frac{1}{2}+\&c.)=2\,(2+\sqrt{2})^2$ $=23.3$ sq. ft.

87. When $x=0$, $y=b$: assume $y=b+Ax+Bx^2+Cx^3+\&c.$: put this value for $y$ in the given expression, and equate to zero coefficients of different powers of $x$.

88*. Let the tolls be supposed *actually* coincident, and be the $x^{\text{th}}$ of the first and $y^{\text{th}}$ of the second bell; then $\dfrac{x}{235}=\dfrac{y}{243}$, or $\dfrac{y}{x}=\dfrac{243}{235}=1+\dfrac{1}{29+}\,\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{2}$, where the last convergent being $\frac{91}{88}$, it follows that the 88th of the first and 91st of the second will be *most nearly* coincident. Let $x$ = distance required from the first bell, expressed in *miles;* then since sound travels 4860 *feet* in a *second*, and $\therefore \frac{4860}{3.1760} \times 60 \times 60 = \frac{81.450}{11}$ *miles* in an *hour*, we have $\frac{87}{235} + \frac{11x}{81.450}$ = time taken for 87 strokes of the first bell + time for 88th stroke to reach the hearer = a similar expression for the second bell $= \frac{90}{243} + \frac{11(3-x)}{81.450}$, whence $x = \frac{1821}{1034}$ of a mile.

89. $\frac{a+x}{ax}\sqrt{(a+x)} = \frac{\sqrt{x}}{b}$, or $\left(\frac{a+x}{x}\right)^{\frac{3}{2}} = \frac{a}{b}$, and $\frac{a+x}{x} = \left(\frac{a}{b}\right)^{\frac{2}{3}}$, &c.

90. (i) Two cases are possible *Ss* or *ss*: $P_1 = \frac{4}{10} = \frac{2}{5}$, $p_1 = \frac{1}{2}$, $P_1p_1 = \frac{1}{5}$, $P_2 = \frac{3}{5}$, $p_2 = 1$, $P_2p_2 = \frac{3}{5}$; $\therefore$ $Q_1 = \frac{1}{4}$, $Q_2 = \frac{3}{4}$, and $C$'s expectation $= \frac{1}{2} \times \frac{1}{4}$ of $21s + \frac{1}{2} \times \frac{3}{4}$ of $2s = 3\frac{3}{8}s$.

In (ii) $P_1$, $P_2$, &c. remain the same; but since $B$ actually receives his coin, a shilling, $C$'s expectation $= \frac{1}{4}$ of $20s + \frac{3}{4}$ of $15s = 5\frac{3}{4}s$.

In (iii) $P_1 = \frac{2}{5}$, $P_2 = \frac{3}{5}$, as before; but $p_1$ is now the compound chance (1) of a shilling being seen and (2) of $B$ receiving one, and, therefore, $= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, $p_2 = 1 \times 1 = 1$; $\therefore$ $P_1p_1 = \frac{1}{10}$, $P_2p_2 = \frac{6}{10}$, $Q_1 = \frac{1}{7}$, $Q_2 = \frac{6}{7}$, and $C$'s expectation $= \frac{1}{7}$ of $20s + \frac{6}{7}$ of $1s = 3\frac{5}{7}s$.

93. $\{1 - x(a-bx)\}^{-\frac{1}{2}} = 1 + \frac{1}{2}x(a-bx) + \frac{3}{8}x^2(a-bx)^2 + \frac{5}{16}x^3(a-bx)^3 + \frac{35}{128}x^4(a-bx)^4 +$ &c.: expand each binomial as far as necessary.

94. See [98 Ex. 2].

95. Sum of squares of $2n+1$ terms $= a^2 . \frac{(r^2)^{2n+1} - 1}{r^2 - 1}$; sum of $2n+1$ terms $= a . \frac{r^{2n+1} - 1}{r - 1}$; sum of $n+1$ *odd* terms, $a + ar^2 +$ &c., $= a . \frac{(r^2)^{n+1} - 1}{r^2 - 1}$, sum of $n$ *even* terms, $ar + ar^3 +$ &c. $= ar . \frac{(r^2)^n - 1}{r^2 - 1}$, and difference of these last two sums

$$= \frac{a}{r^2-1} . \{r^{2n+2} - r^{2n+1} + r - 1\} = \frac{a}{r^2-1} . (r^{2n+1}+1)(r-1) = a . \frac{r^{2n+1} + 1}{r+1},$$

whence the result plainly follows.

96. $\sqrt{(xy)} : \frac{2xy}{x+y} = m : n$, whence, (if $z = \frac{x}{y}$,) $nz - 2m\sqrt{z} + n = 0$

$\therefore \sqrt{z} = \frac{m+\sqrt{(m^2-n^2)}}{n}$, and $\frac{x}{y} = \frac{\{m+\sqrt{(m^2-n^2)}\}^2}{n^2} = \frac{m+\sqrt{(m^2-n^2)}}{m-\sqrt{(m^2-n^2)}}$.

97. $p_0 r^{n-1}$ will occur $\lfloor\underline{n-1}$ times, since the other $n-1$ digits may be so often permuted; similarly for $p_0 r^{n-2}$, &c. and for $p_1 r^{n-1}$, $p_1 r^{n-2}$, &c. $p_2 r^{n-1}$, $p_2 r^{n-2}$, &c.: $\therefore$ whole sum

$$= \lfloor\underline{n-1}\, p_0 (r^{n-1} + r^{n-2} + \text{\&c.}) + \lfloor\underline{n-1}\, p_1 (r^{n-1} + r^{n-2} + \text{\&c.}) + \text{\&c.}$$

$$= \lfloor\underline{n-1}\, (p_0 + p_1 + \text{\&c.}) \frac{r^n - 1}{r - 1}:$$

its greatest value will be when $p_0$, $p_1$, &c. include all the digits admissible in the scale of $r$, in which case $n = r$, and $p_0 + p_1 + p_2 + \text{\&c.} = 0 + 1 + 2 + \text{\&c.} = \frac{1}{2}r(r-1)$.

98. $R$, $r$, the radii, $W$, $w$, the weights of corresponding spheres, if solid; $\therefore$ $W - w$ = weight of shell, and $\frac{W}{w} = \frac{R^3}{r^3}$, or $\frac{W-w}{W} = \frac{R^3 - r^3}{R^3}$; $\therefore \frac{7}{8} = 1 - \left(\frac{r}{R}\right)^3$, and $\frac{r}{R} = \frac{1}{2}$;

also $\frac{r'}{R} = \frac{\frac{3}{2}r}{R} = \frac{3}{4}$; $\therefore \frac{R^3 - r^3}{R^3} = \frac{37}{64}$, and $\therefore \frac{R^3 - r^3}{R^3 - r^3} = \frac{56}{37}$.

99. $x + \sqrt{x} - \sqrt{(x^2 - x)} = a\sqrt{x}$; $\therefore \sqrt{x} = 0$, and $\sqrt{x} + 1 - \sqrt{(x-1)} = a$;

hence $a - 1 = \sqrt{x} - \sqrt{(x-1)} = \frac{x - (x-1)}{\sqrt{x} + \sqrt{(x-1)}} = \frac{1}{\sqrt{x} + \sqrt{(x-1)}}$,

or $\frac{1}{a-1} = \sqrt{x} + \sqrt{(x-1)}$; $\therefore a - 1 + \frac{1}{a-1} = 2\sqrt{x}$.

100. Two cases are possible; it fell (1) *heads*, (2) *tails*, and $P_1 = \frac{1}{2} = P_2$: if (1) be the true event, $A$ and $B$ must have spoken truly, $C$ falsely, and $p_1 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5}$; so $p_2 = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{4}{5}$; &c.

101. (i) The two equations must have a common factor, $x - m$, suppose: we may assume then $ax^2 + bx + c = (x - m)\left(ax - \frac{c}{m}\right)$, whence $am + \frac{c}{m} = b$, and similarly, $a'm + \frac{c'}{m} = b'$; from which equations we get $(ac' - a'c)\,m = bc' - b'c$, and $\frac{ac' - a'c}{m} = ab' - a'b$; multiply these together.

(ii) One of them must be a multiple of the other; put $ax^2+bx+c=m(a'x^2+b'x+c')$; $\therefore$ $a=ma'$, $b=mb'$, $c=mc'$; or $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'}$. 102. $\frac{1}{2}n(n+1)+\frac{1}{2}(n+1)(n+2)=$ &c.

103. Put $b=ar$, $c=br=ar^2$, &c.: then

$$S=\frac{1}{a^3+b^3}+\frac{1}{(a^3+b^3)r^3}+\frac{1}{(a^3+b^3)r^6}+\text{\&c.}$$
$$=\frac{1}{a^3+b^3}\{1+\frac{1}{r^3}+\text{\&c.}\}=\frac{1}{a^3+b^3}\{1+\frac{a^3}{b^3}+\frac{a^6}{b^6}+\text{\&c.}\}=\text{\&c.}$$

104. See [27].

106. Width of $S$ : width of $s=m:n$; $\therefore$ area of $S$ : area of $s=m^2:n^2$; again, thickness of $S$ : thickness of $s=p:q$; $\therefore$ bulk of $S$ : bulk of $s=m^2p:n^2q$; $\therefore$ n° of *shillings* in given bulk : n° of *sovereigns* $=m^2p:n^2q$, and value of former : value of latter $=m^2p:20n^2q$.

107. $S_1+S_2+\text{\&c.}=\frac{1}{2}n(n+1)+\frac{1}{2}n(3n+1)+\text{\&c.}=\frac{1}{2}n\{(1+3+\text{\&c.})n+m\}$ $=\frac{1}{2}n\{m^2n+m\}$.

108. As in [139], $10a=bq_1+r_1$, $10r_1=bq_2+r_2$, &c., $10r_n=bq_n+a$; $\therefore$, adding, $9(a+r_1+r_2+\text{\&c.}+r_n)=b(q_1+q_2+\text{\&c.}+q_n)$: hence, since $b$ is prime to 3, we shall have $q_1+q_2+$ &c., or the *sum of the digits* of the period, and $\therefore$ [130] the period itself, divisible by 9, and then $a+r_1+r_2+\text{\&c.}+r_n$ must also be divisible by $b$.

109. (i) $\dfrac{a+x}{\sqrt{(a^2-x^2)}}=\sqrt{\dfrac{a+x}{a-x}}=\dfrac{b+x}{b-x}$; square, &c.

(ii) Put $\dfrac{x^2-a^2}{cx}=z$; then $z+\dfrac{1}{z}=b$, &c.

110. See [Ex. **49** 12]. Here $P_1=\frac{1}{5}$, $P_2=\frac{4}{5}$, the *a priori* chances of my winning or losing: if I won, $p_1=$ chance of a true report reaching me $=abc+ab'c'+a'bc'+ab'c'=\frac{11}{20}$, $p_2=\frac{9}{20}$; &c.

112. $x+\dfrac{m}{nx}-1-\dfrac{m}{n}=(x-1)(1-\dfrac{m}{nx})=\dfrac{1}{x}(x-1)(x-\dfrac{m}{n})$, which is *positive*, if $x >$ or $<$ each of 1 and $\dfrac{m}{n}$, but *negative*, if $x$ lies between them.

115. Find coefficient of $x^{13}$ in $x^5(a-bx^2+\text{\&c.})^5$, or of $x^8$ in

$$(a-bx^2+cx^4-dx^6+ex^8-\text{\&c.})^5,$$

where, as in [99], we shall have

$$\alpha+\beta+\gamma+\delta+\varepsilon=5,\quad 2\beta+4\gamma+6\delta+8\varepsilon=8,$$

it being useless to go beyond $ex^8$, because $fx^{10}$ would not affect the coefficient of $x^8$.

116. Put $N = m + x$, $n = m - x$; then

$$\sqrt{\frac{N}{n}} = \frac{\sqrt{(m^2 - x^2)}}{m - x} = \frac{m}{m - x}\{1 - \frac{x^2}{2m^2} - \frac{x^4}{8m^4} - \&c.\} = \frac{2m^2 - x^2}{2m\,(m - x)}$$

nearly, which $= \frac{m + x}{2m} + \frac{1}{4} \cdot \frac{2m}{m - x} = \frac{N}{N + n} + \frac{1}{4} \cdot \frac{N + n}{n}$.

The same reasoning, of course, holds, if $N < n$, or the sign of $x$ be changed.

117. Here $\frac{1}{4} \cdot \frac{N + n}{n} - \frac{N}{N + n}$, or, by Ex. 116, $\frac{x^2}{2m\,(m - x)} < \frac{1}{10^p}$;

$\therefore$ *a fortiori*, $\frac{x^2}{2m^2} < \frac{1}{10^p}$: now the first term omitted of the series for $\sqrt{\frac{N}{n}}$ is $\frac{x^4}{8m^3\,(m - x)}$, which is $\therefore < \frac{1}{10^p} \cdot \frac{x^2}{4m^2} < \frac{1}{10^{2p}}$; so the next term $< \frac{1}{10^{3p}}$, &c.; hence the result will be correct for the first $2p$ decimal places.

Ex. $\sqrt{30} = \frac{11}{2}\sqrt{\frac{120}{121}}$: now if $N = 120$, $n = 121$, then

$$\frac{N}{N + n} \sim \frac{1}{4} \cdot \frac{N + n}{n} = \frac{120}{241} \sim \frac{241}{484} = \frac{1}{116644} = .000008 \;\&c.:$$

hence the two quantities must have at least their first *four* digits the same, and $\sqrt{\frac{120}{121}}$ will be given correct to *eight* places by the formula.

118. We have given $n = a + (m - 1)\,d$, $m = a + (n - 1)\,d$, whence $d = -1$, $a = m + n - 1$: putting these for $a$ and $d$ in

$$S = \{2a + (x - 1)\,d\}\,\tfrac{1}{2}x = \tfrac{1}{2}\,(m + n)\,(m + n - 1),$$

we get $x^2 - \{2\,(m + n) - 1\}\,x + (m + n)\,(m + n - 1) = 0$, whence $x = m + n$ or $m + n - 1$. Use now these values for $a$, $d$, $x$, in $l = a + (x - 1)\,d$.

119. Sum (1) as a G. P.; then $1 \div (1 - \frac{x^2}{5y^2}) = \frac{5x^2}{y^2}$, whence (by a common Quadratic) $\frac{x^2}{y^2} = \frac{5 \pm \sqrt{21}}{2}$, and $\therefore \frac{x}{y} = \pm\frac{\sqrt{7} \pm \sqrt{3}}{2}$: Also (2) is the expansion of $\{1 - (x + y)\}^{-\frac{1}{2}} = \sqrt{1\tfrac{1}{4}}$, or $x + y = \frac{1}{5}$; now, by the former result, $\frac{x}{x + y} = \frac{\sqrt{7} \pm \sqrt{3}}{\sqrt{7} \pm \sqrt{3} \pm \sqrt{2}}$;

$$\therefore x = \frac{\sqrt{7} \pm \sqrt{3}}{5\,(\sqrt{7} \pm \sqrt{3} \pm \sqrt{2})} = \frac{(\sqrt{7} \pm \sqrt{3})\,(\sqrt{7} \mp \sqrt{3} \pm 2)}{5\,\{(\sqrt{7} \pm 2)^2 - 3\}} = \frac{2 \pm \sqrt{7} \pm \sqrt{3}}{10\,(2 \pm \sqrt{7})}$$

$$= \frac{3 \pm \sqrt{21} \pm 2\sqrt{3}}{30};\; y = \tfrac{1}{5} - x = \&c.$$

120. (i) $A$ has *four* drawings, $B$ *three;* the chance of success in each is $\frac{1}{7}$, for if *seven* persons had each a right to draw, it would not matter *in what order* they drew: $\therefore$ $A$'s exp. is $\frac{1}{7}$ of $7s = 4s$, $B$'s $3s$.

(ii) $B$'s stake ought to have been to $A$'s :: 3 : 4.

121. (ii) $2^2.5.7x = y^3$; hence, to make the first side a cube, $x$ must be of the form $2.5^2.7^2.m^3$. (iii) $xy = 2^{2n}.5^{2n}$: see [199].

122. (i) $e^{-x} = \dfrac{1}{\sqrt{(1+y^2)}+y} = \dfrac{\sqrt{(1+y^2)}-y}{(1+y^2)-y^2} = \sqrt{(1+y^2)} - y$; &c.

(ii) $\{\sqrt{\frac{1}{2}a} + \sqrt{(c-\frac{1}{2}a)}\} + \{\sqrt{\frac{1}{2}a} - \sqrt{(c-\frac{1}{2}a)}\} =$ &c.

124. $a^2b^2\{a^2-(a^2-b^2)x^2\}^{-\frac{3}{2}} = a^2(a^2-2ac)(a^2-2acx^2)^{-\frac{3}{2}}$

$$= a^3(a-2c)\,a^{-3}\left(1+\frac{3c}{a}x^2+\text{\&c.}\right) = \text{\&c.}$$

126. Put $y = \dfrac{1}{x}$, so that the values of $y$ will be $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$; then $x = \dfrac{1}{y}$, and, putting this for $x$, the equation becomes

$$y^3 - \tfrac{3}{4}y^2 + \tfrac{1}{2}y - \tfrac{1}{4} = 0,$$

where [58] $\frac{3}{4} =$ sum of roots $= \alpha^{-1} + \beta^{-1} + \gamma^{-1}$.

(ii) $(x^3+3x)^2 = (2x^2+4)^2$, whence $x^6 + 2x^4 - 7x^2 - 16 = 0$; put $y = x^2$, then $y^3 + 2y^2 - 7y - 16 = 0$, where the values of $y$ are $\alpha^2, \beta^2, \gamma^2$, &c.

(iii) In $y^3+2y^2-7y-16=0$, put $z = \dfrac{1}{y}$, and reason as in (i).

127. Let value of diamonds $= p\,(\text{weight})^2$, and that of rubies $= q\,(\text{weight})^{\frac{3}{2}}$; then (1) $pa^2 = mqb^{\frac{3}{2}}$, (2) $pa^2 + mqb^{\frac{3}{2}} = c$, whence $p = \dfrac{cm}{(m+1)\,a^2}$, $q = \dfrac{c}{(m+1)\,b^{\frac{3}{2}}}$; and the values required are $px^2$, $qx^{\frac{3}{2}}$.

128. Wine in $A$ after *first* operation $= a - c$, in $B = c$; $\therefore$ wine in $A$ after *second* $= a - c - \left(\dfrac{a-c}{a} \times c\right) + \left(\dfrac{c}{b} \times c\right)$, which $= a - c$, if $\dfrac{c}{b} = \dfrac{a-c}{a}$, or $c = \dfrac{ab}{a+b}$.

129. (i) $\dfrac{x+a}{x+b} = \left\{\dfrac{x+\frac{1}{2}(a+2c-b)}{x+c}\right\}^2 = \dfrac{x^2+(a+2c-b)\,x+\frac{1}{4}(a+2c-b)^2}{x^2+2cx+c^2}$;

whence, getting rid of fractions, the terms in $x^3$ and $x^2$ cancel, and

$$x(a^2+2ab-3b^2-4ac+4bc)=x\{(a+3b)(a-b)-4c(a-b)\}$$
$$=-(a-b)^2b-4bc(a-b)+(a-b)4c^2;$$
$$\therefore x(a+3b-4c)=(b-2c)^2-ab, \text{ \&c.}$$

(ii) Assume $x^4-2x^3+\frac{2}{c^2}x-\frac{1}{c^2}=0=(x^2-x+p)^2-(qx+r)^2$; then, equating coeff^s, $q^2=2p+1$, $-qr=p+\frac{1}{c^2}$, $r^2=p^2+\frac{1}{c^2}$; and $\therefore (2p+1)(p^2+\frac{1}{c^2})=q^2r^2=(p+\frac{1}{c^2})^2$, whence

$$p=\sqrt[3]{\frac{1-c^2}{2c^4}},\ q^2=1+2\sqrt[3]{\frac{1-c^2}{2c^4}},\ r^2=\frac{1}{c^2}+\left(\frac{1-c^2}{2c^4}\right)^{\frac{2}{3}};$$

then $x^2-x+p=\pm(qx+r)$, &c.

130. Find the chances for *A*'s and *B*'s events ($\frac{1}{6}$, $\frac{1}{12}$) [219 Ex. 1], by help of the coefficients of $x^7$ and $x^4$ in $(x+x^2+\text{\&c.})^2$, or of $x^5$ and $x^2$ in $(1+x+\text{\&c.})^2=\left(\frac{1-x^6}{1-x}\right)^2$, or in $\frac{1}{(1-x)^2}$, since these coefficients cannot be affected by leaving out the $x^6$; hence *A*'s chance of success on each trial is double of *B*'s, and also *B*'s exp. in a *single* cast = $\frac{1}{12}$ of 3*s* – $\frac{1}{6}$ of 1*s* = 1*d.* But the question asks for *B*'s exp. on the *long run*, the throwing being continued till one or the other wins. Now if $x=B$'s chance on this supposition, or his chance of success *compared with A*'s, then plainly, from the foregoing reasoning, $2x=A$'s, and $x+2x=$ certainty $=1$; $\therefore x=\frac{1}{3}$, and *B*'s expectation = $\frac{1}{3}$ of 3*s* – $\frac{2}{3}$ of 1*s* = 4*d.*

131. See the solution of Ex. 122.

132. (i) Assume $\sqrt{(2n\sqrt{-1})}=\sqrt{x}+\sqrt{y}\sqrt{-1}$; then

$$2n\sqrt{-1}=x+2\sqrt{(xy)}\sqrt{-1}-y;\ \therefore x-y=0,\ \sqrt{(xy)}=n,$$

whence $x=y=\sqrt{n}$, &c.

In (ii) obtain similarly $\sqrt{(4\pm3\sqrt{-20})}=3\pm\sqrt{-5}$, and add them.

133. log $1\frac{2}{5}$ or log 1.4 gives log 14 or log 2 + log 7; log 144 gives log 12 or 2 log 2 + log 3; log 441 gives log 21 or log 3 + log 7; by means of these *data* we get log 2, log 3, and log 7, and thus (observing also that log 5 = 1 – log 2) we may get all the logs required.

134. (i) $z^2 - a^2 = 2xy,\ b^3 = z(x^2 - xy + y^2) = z\{a^2 - \frac{1}{2}(z^2 - a^2)\}$.

(ii) $a^7 = x^7 + y^7 + 7xy\,(x^5 + y^5) + 21x^2y^2\,(x^3 + y^3) + 35x^3y^3\,(x + y)$

$= b^7 + 7az^2\{x^4 + y^4 - xy\,(x^2 + y^2) + x^2y^2\} + 21az^4\,(x^2 - xy + y^2) + 35az^6$

$\therefore a^7 - b^7 = 7az^2\{a^4 - 5xy\,(x^2 + y^2) - 5x^2y^2\} + 21az^4\,(a^2 - 3z^2) + 35az^6$

$= 7az^2\{a^4 - 5z^2\,(a^2 - 2z^2) - 5z^4 + 3z^2\,(a^2 - 3z^2) + 5z^4\} =$ &c.

136. $b$ is the last of $n + 2$ terms, of which $a$ and $a$ are the first and second; $\therefore$ the common ratio being $\frac{a}{a}$, we have $b = a\left(\frac{a}{a}\right)^{n+1}$.

137. Wine in $B$ after second operation $= c + \frac{a - c}{a} \times c - \frac{c}{b} \times c$

$= c\,(1 + p)$, $\left(\text{if } p = 1 - c\,\frac{a + b}{ab}, \text{ or } c = \frac{ab}{a + b}\,(1 - p),\right) = \frac{ab}{a + b}\,(1 - p^2)$.

Assume the Law for $(n - 1)$th operation, so that wine in $B$ after it $= \frac{ab}{a + b}\,(1 - p^{n-1})$, and $\therefore$ wine remaining in $A$ $= a - \frac{ab}{a + b}\,(1 - p^{n-1})$; then wine in $B$ after $n$th operation

$= \frac{ab}{a + b}\,(1 - p^{n-1}) + \frac{c}{a}\left\{a - \frac{ab}{a + b}\,(1 - p^{n-1})\right\} - \frac{c}{b} \cdot \frac{ab}{a + b}\,(1 - p^{n-1})$

$= \frac{ab}{a + b}\,\{1 - p^{n-1} + (1 - p)\,p^{n-1}\} = \frac{ab}{a + b}\,(1 - p^n)$.

138. Let $AB$, $BC$, $CA$, be the outer lines of the wedge, containing $n$ men each; then, if the wedge were *solid*, the number of men in it would be $1 + 2 + 3 +$ &c. $+\, n = \frac{1}{2}n\,(n + 1) = S$. Now let $DE$ be drawn, the $(n - r)$th line from the vertex $A$, and $\therefore$ containing $n - r$ men: from each end of $DE$ cut off $r$ men; then the remaining line of $n - 3r$ men is the base of the inner wedge, which must be removed to leave the hollow one required: then

$1 + 2 + 3 +$ &c. $+\, (n - 3r) = \frac{1}{2}\,(n - 3r)\,(n - 3r + 1) = S'$;

and number required $= S - S'$.

139. From (1) $\left(\frac{x^2}{y^2} + \frac{2x}{\sqrt{y}} + y\right) + \left(\frac{x}{y} + \sqrt{y}\right) + \frac{1}{4} = \frac{81}{4}$, and therefore $\frac{x}{y} + \sqrt{y} = 4$ or $-5$; hence, by (2), $\frac{4y - 8}{y} + \sqrt{y} = 4$ or $-5$, whence (i) $y^{\frac{3}{2}} = 8$, (ii) $y^{\frac{3}{2}} + 9y = 8$; from (i) $y^{\frac{1}{2}} = 2$, &c.: in (ii) $y^{\frac{1}{2}} = -1$ is plainly a root; divide $y^{\frac{3}{2}} + 9y - 8$ by $y^{\frac{1}{2}} + 1$, &c.: then we shall get $y^{\frac{1}{2}} = -4 \pm 2\sqrt{6}$, &c.

140. In Ex. 130 we found that $A$'s chance of success in *each* trial is $\frac{1}{6}$, $B$'s $\frac{1}{12}$, so that $A$'s is double of $B$'s.

Let $x = B$'s chance upon the *long run*, if $B$ has the first cast; then, plainly, $2x = A$'s, if $A$ has the first cast: but in (i) $B$ is to have the second throw; $\therefore$ his actual chance of success in the long run $=$ (chance of $A$'s failing the first time) $\times\, x = \frac{5}{6}x$; and $A$'s total chance $+$ $B$'s total chance $= 1$, or $2x + \frac{5}{6}x = 1$.

In (ii) $A$'s total chance $= \frac{11}{12} \times 2x$, and $B$'s $= x$; $\therefore x + \frac{11}{6}x = 1$. Or otherwise thus: In (i) chance of $B$ *having* a first throw $=$ chance of $A$'s failing $= \frac{5}{6}$; $\therefore$ chance of $B$'s success on his first trial $= \frac{5}{6} \times \frac{1}{12}$: so chance of $B$ *having* a second throw is contingent upon $A$ failing once, $B$ failing once, and $A$ failing again, and $\therefore = \frac{5}{6} \times \frac{11}{12} \times \frac{5}{6}$; $\therefore$ chance of $B$'s success on a second trial $= \frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{1}{12}$; and so on: hence $B$'s total chance of success $= \frac{5}{72}\{1 + \frac{55}{72} + (\frac{55}{72})^2 + \&c.\}$ $= \dfrac{5}{72} \times \dfrac{1}{1 - \frac{55}{72}} = \dfrac{5}{17}$; and $A$'s $= 1 - \frac{5}{17} = \frac{12}{17}$. Similarly in (ii).

141. $c^2 = a^2 + b^2 - 2a^2b^2 + 2ab\sqrt{(1-a^2)(1-b^2)}$: $\therefore \{(a+b)^2 - c^2\}\{c^2 - (a-b)^2\}$
$= \{2ab + 2a^2b^2 - 2ab\sqrt{(1-a^2)(1-b^2)}\} \times \{2ab - 2a^2b^2 + 2ab\sqrt{(1-a^2)(1-b^2)}\}$
$= 4a^2b^2\,[1 + \{ab - \sqrt{(1-a^2)(1-b^2)}\}]\,[1 - \{ab - \sqrt{(1-a^2)(1-b^2)}\}]$
$= 4a^2b^2\,\{a^2 + b^2 - 2a^2b^2 + 2ab\sqrt{(1-a^2)(1-b^2)}\} = 4a^2b^2c^2$.

142. (i) $S = 1 - n + \dfrac{n(n+1)}{1.2} - \&c. - n + n(n+1) - \dfrac{n(n+1)(n+2)}{1.2} + \&c.$

$$= (1+1)^{-n} - n\,(1+1)^{-(n+1)} = \&c.$$

(ii) Neglecting its sign, the $r^{\text{th}}$ term, $P_r$,

$$= r\,\frac{n(n+1)\ldots(n+r-2)}{1.2 \quad \ldots \quad (r-1)},$$

and $\dfrac{P_r}{P_{r-1}} = \dfrac{r}{r-1} \cdot \dfrac{n+r-2}{r-1} = 1 + \dfrac{nr-1}{(r-1)^2}$, which is $\gtrless 1$, as $nr \gtrless 1$.

144. The roots of $x^2 + px + q = 0$, being equal, must each be $-\frac{1}{2}p$, and we have also $\frac{1}{4}p^2 - q = 0$: but the second equation is

$$a\,(x^2 + px + q) + b\,(px + 2q) = 0,$$

which is satisfied by $x = -\frac{1}{2}p$, since then $x^2 + px + q = 0$, and also $px + 2q = -\frac{1}{2}p^2 + 2q = 0$. Hence, if $\beta$ be the other root, $\frac{1}{2}p - \beta$ is the coeff. of $x$ in $x^2 + p\,(1 + \dfrac{b}{a})\,x + q\,(1 + \dfrac{2b}{a}) = 0$.

145. Assume for the parts $a-3d,\ a-d,\ a+d,\ a+3d$.

146. By [43]
$$\begin{array}{rl}(a_1b_1+a_2b_2+\&c.)^2=a_1^2b_1^2+ & 2a_1b_1a_2b_2+\&c.\\ + & a_2^2b_2^2+\&c.\\ & \qquad\quad +\&c.\end{array}$$

and $\therefore$ is
$$\begin{array}{llll}<a_1^2b_1^2+(a_1^2b_2^2+a_2^2b_1^2)+\&c. & & & <a_1^2(b_1^2+b_2^2+\&c.)\\ \qquad + \qquad a_2^2b_2^2 & +\&c. & & +a_2^2(b_1^2+b_2^2+\&c.)\\ & +\&c. & & +\&c.\end{array}$$

that is, is $<(a_1^2+a_2^2+\&c.)\,(b_1^2+b_2^2+\&c.)$,

unless $a_1:b_1=a_2:b_2=\&c.$, when $a_1b_2=a_2b_1$ and $2a_1b_1a_2b_2=a_1^2b_2^2+a_2^2b_1^2$, &c.

147. $$e^{-x}=1+\frac{(-x)}{1}+\frac{(-x)^2}{1.2}+\&c.=1-\frac{x}{1}+\frac{x^2}{1.2}-\&c.+(-1)^{n-2}\frac{x^{n-2}}{\lfloor\underline{n-2}}$$
$$+(-1)^{n-1}\frac{x^{n-1}}{\lfloor\underline{n-1}}+(-1)^n\frac{x^n}{\lfloor\underline{n}}+\&c.$$

148. Let $x$ be the sum laid out; then $\frac{x}{a}$ = n° of *cents* assured by it, and $\frac{rx}{a}$ = Int. on this sum, which, by Question, = reduced income $= A-x$.

149. (i) $n\{(1+x)-1\}=\{\sqrt{(1+x)}-1\}\{\sqrt{(1-x)}+1\}$, or $n\{\sqrt{(1+x)}+1\}=\sqrt{(1-x)}+1$: $\therefore n\sqrt{(1+x)}-\sqrt{(1-x)}=1-n$, whence, squaring, &c. $2n\sqrt{(1-x^2)}=2n+(n^2-1)\,x$; square again, &c.

(ii) $x^2-y^2=a\,(x-y)-b\,(x-y)$, whence we get $x-y=0$, or $x+y=a-b$; $x-y=0$, or $y=x$, gives by (1) $x=0$ or $a+b$; and $y=a-b-x$ gives by (1) $x^2-(a-b)\,x-b\,(a-b)=0$, &c.

150. (i) Chance of dipping into bag $A$ or $B$ is $\frac{1}{2}$; $\therefore$ chance of drawing white from $A$ is $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$, and from $B$, $\frac{1}{2}\times\frac{3}{4}=\frac{3}{8}$; then $\frac{1}{6}+\frac{3}{8}=\frac{13}{24}$.

(ii) Chance of drawing *white* any time is by (i) $\frac{13}{24}$; $\therefore$ chance of drawing *black* is $\frac{11}{24}$; hence chance of drawing white the first time and black the second $=\frac{143}{576}$ = chance of drawing black the first and white the second; $\therefore$ chance required $=\frac{143}{288}$.

(iii) Since there is only *one* white in $A$, we cannot go to it more than once in the drawing of three white balls, so that the drawings can only be out of $ABB$, $BAB$, $BBA$, or $BBB$. The chance for each of the first three of these is the same, viz. $(\frac{1}{2}\times\frac{1}{3})\times(\frac{1}{2}\times\frac{3}{4})\times(\frac{1}{2}\times\frac{2}{3})=\frac{1}{48}$, (where observe that the factor $\frac{1}{2}$ expresses the chance of dipping into the corresponding bag;) and that of the last is, similarly, $\frac{1}{32}$; and $\frac{3}{48}+\frac{1}{32}=\frac{3}{32}$.

151. Let $x$ be one part; then $x\{(2n+1)-x\}=m$, or

$$x=\tfrac{1}{2}\left[2n+1\pm\sqrt{\{(2n+1)^2-4m\}}\right],$$

where the greatest value $m$ can have is when $4m=4(n^2+n)$, and $x=\frac{1}{2}\{(2n+1)\pm 1\}=n$ or $n+1$.

152. $(x+1)^n=(1+x)^n=1+C_1x+C_2x^2+\&c$, $(x-1)^n=x^n-C_1x^{n-1}+C_2x^{n-2}-\&c$; $\therefore (x+1)^n\times(x-1)^n$ or $(x^2-1)^n=\&c.$ by Multiplication: equate coefficients of $x^n$, and see [97 Ex. 6].

153*. Eliminating, we get $p+q=\dfrac{2(ab-cd)}{a+b-c-d}$, $pq=\dfrac{ab(c+d)-cd(a+b)}{a+b-c-d}$;

$$\therefore \left(\frac{p+q}{2}\right)^2-pq=\left(\frac{p-q}{2}\right)^2=\frac{(ab-cd)^2-\{ab(c+d)-cd(a+b)\}\{a+b-c-d\}}{(a+b-c-d)^2},$$

where the numerator

$$=a^2b^2-2abcd+c^2d^2-[a\{b(c+d)-cd\}-bcd]\{a+b-(c+d)\}$$
$$=a^2\{b^2-b(c+d)+cd\}-a\{b^2(c+d)-b(c+d)^2+cd(c+d)\}+c^2d^2+b^2cd-bcd(c+d)$$
$$=a^2(b-c)(b-d)-a(c+d)(b-c)(b-d)+cd(b-c)(b-d)$$
$$=\{a^2-a(c+d)+cd\}(b-c)(b-d)=(a-c)(a-d)(b-c)(b-d).$$

154. $ar^m+ar^n>ar^{m+p}+ar^{n-p}$, if $r^m(1-r^p)>r^{n-p}(1-r^p)$, that is, (when $r<1$) if $r^m>r^{n-p}$, or (when $r>1$) if $r^m<r^{n-p}$, either of which is the case, since $n-p>m$.

155. (ii) Here the $r^{\text{th}}$ coeff. is greatest for the first value of $r$ which makes $\dfrac{n-r+1}{r}$, (the factor by which the $(r+1)^{\text{th}}$ coeff. differs from the $r^{\text{th}}$,) $<1$;

$$\therefore \frac{n+1}{r}-1<1, \text{ or } r>\frac{n+1}{2}>\frac{11\sqrt{2}+2}{4}>\frac{11(1.414\,\&c.)+2}{4}.$$

156. $(a^2+b^2)(c^2+d^2)=(ac\pm bd)^2+(ad\mp bc)^2$, that is, the product of $a^2+b^2$ and $c^2+d^2$ can be expressed in the form $a'^2+b'^2$ in two ways: again $(a^2+b^2)(c^2+d^2)(e^2+f^2)=(a'^2+b'^2)(e^2+f^2)$, which may be similarly resolved in two ways, that is, since $a'$ and $b'$ have themselves *double* values, the product of $a^2+b^2$, $c^2+d^2$, and $e^2+f^2$, can be expressed in the form $a''^2+b''^2$ in $2^2$ ways; and so on.

157. Let $a_1$, $a_2$, &c. be the first terms of the series respectively:

$$\text{then }\ \{2a_1 + (n-1)\,1\}\,\tfrac{1}{2}n = na_1 + \tfrac{1}{2}n\,(n-1) = n^2,$$

$$\text{or }\ a_1 = n - \tfrac{1}{2}(n-1) = \tfrac{1}{2}(n+1);$$

$$\text{so }\ a_2 = n - (n-1),\ \ a_3 = n - \tfrac{3}{2}(n-1),\ \&\text{c.}$$

158. Let $a^r + (a+b)^r + \&\text{c} + \{a + (n-1)\,b\}^r$

$$= An + Bn^2 + \&\text{c} + Qn^r + Rn^{r+1} + \&\text{c};$$

then $a^r + (a+b)^r + \&\text{c.} + \{a + nb\}^r$

$$= A\,(n+1) + B\,(n+1)^2 + \&\text{c} + Q\,(n+1)^r + R\,(n+1)^{r+1} + \&\text{c};$$

and, subtracting,

$$(a+nb)^r = A + B\,(2n+1) + C\,(3n^2+3n+1) + \&\text{c.}$$
$$+ Q\,\{(n+1)^r - n^r\} + R\,\{n+1)^{r+1} - n^{r+1}\},$$

it being useless to go beyond $R$, since we should have to equate to *zero* the coeff. of each power of $n$ higher than $n^r$: hence, equating coeff$^s$ of $n^r$, we have $b^r = R\,(r+1)$, and

$$\therefore\ S = An + Bn^2 + \&\text{c} + \frac{b^r n^{r+1}}{r+1};$$

$$\therefore\ \frac{S}{n^{r+1}} = \frac{An + Bn^2 + \&\text{c.}}{n^{r+1}} + \frac{b^r}{r+1} = \frac{b^r}{r+1},\ \text{when } n \text{ is great.}$$

159*. (i) Getting rid of fractions, we have

$$\sqrt{(1+a^2)}\,(1+ax) = 2a\,\sqrt{(1+x^2)};$$

squaring, we get $a^2x^2\,(a^2-3) + 2ax\,(1+a^2) + (1-3a^2) = 0$.

(ii) Put $y$ for $x-a$, $z$ for $x-b$; then (85)

$$\frac{2x-(a+b)}{-3\,(a-b)} = \frac{y^3+z^3}{y^3-z^3} = \frac{(y+z)\,(y^2-yz+z^2)}{(y-z)\,(y^2+yz+z^2)} = \frac{\{2x-(a+b)\}\,(y^2-yz+z^2)}{-(a-b)\,(y^2+yz+z^2)};$$

$\therefore\ 2x - (a+b) = 0$, or $x = \frac{1}{2}(a+b)$. [We get no root, it will be found, by putting $3\,(y^2 - yz + z^2) = y^2 + yz + z^2$.]

160. $P_1 = \frac{1}{2} = P_2$, and [218 Cor.] may be neglected in both cases.

Now (i) $p_1 = \frac{1}{3} = \frac{4}{12}$, $p_2 = \frac{3}{4} = \frac{9}{12}$; $\therefore\ Q_1 = \frac{4}{13}$:

(ii) $p_1 = \frac{1}{3}\left(\frac{1}{2}\times 1 + \frac{1}{2}\times 4\right) = \frac{5}{24}$, $p_2 = \frac{3}{4}\left(\frac{1}{2}\times\frac{2}{3} + \frac{1}{2}\times\frac{1}{3}\right) = \frac{9}{24}$; $\therefore\ Q_1 = \frac{5}{14}$.

161. By [130], $N$ and $N'$, since the sum of their digits is the same, will, when divided by 9, leave the same remainder, that is, they must be of the form $N = 9n + r$, $N' = 9n' + r$; $\therefore\ N \sim N' = 9\,(n \sim n')$.

162. Here $\alpha + \beta = 1$, $\alpha\beta = 1$;

$$\therefore\ \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = 1 - 3 = -2.$$

163. Let $\sqrt{n} = a + x$; $\therefore\ n - a^2 = b = 2ax$ nearly, and $x = \dfrac{b}{2a}$;

but $b = 2ax + x^2$ more nearly, or $= x(2a + x) = x\left(2a + \dfrac{b}{2a}\right)$;

$$\therefore\ x = \frac{2ab}{4a^2 + b}.$$

164. (i) $\dfrac{a}{3(x-a)}\log_e\left(1 + \dfrac{x-a}{a}\right) = \dfrac{a}{3(x-a)}\left\{\dfrac{x-a}{a} - \dfrac{1}{2}\left(\dfrac{x-a}{a}\right)^2 + \&c.\right\}$

$= \frac{1}{3}$, when $x = a$.

166. The number of digits being in each case *odd*, $2n + 1$ suppose, we may write the general form for these numbers,

$$N = r^{2n} + 2r^{2n-1} + 3r^{2n-2} + \&c. + 3r^2 + 2r + 1;$$

$$\therefore\ \frac{N}{r} = \qquad r^{2n-1} + 2r^{2n-2} + \&c. + 4r^2 + 3r + 2 + \frac{1}{r};$$

$$\text{or } N\left(1 - \frac{1}{r}\right) = r^{2n} + r^{2n-1} + \&c. + r^n - \left(r^{n-1} + r^{n-2} + \&c. + 1 + \frac{1}{r}\right);$$

$$\therefore\ N \cdot \frac{r-1}{r} = r^n \cdot \frac{r^{n+1}-1}{r-1} - \frac{1}{r} \cdot \frac{r^{n+1}-1}{r-1} = \frac{(r^{n+1}-1)^2}{r(r-1)};$$

$$\text{or } N = \left(\frac{r^{n+1}-1}{r-1}\right)^2 = (r^n + r^{n-1} + \&c. + r^2 + r + 1)^2.$$

167*. Let $A$, $B$, $C$, take respectively $x$, $y$, $z$, days to do the work: then in one day, $B$ and $C$ do $m$ times as much as $A$;

$\therefore$ (1) $\dfrac{1}{y} + \dfrac{1}{z} = \dfrac{m}{x}$; similarly, (2) $\dfrac{1}{x} + \dfrac{1}{z} = \dfrac{n}{y}$, (3) $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{p}{z}$;

from (1) and (2), $\dfrac{1}{y} - \dfrac{1}{x} = \dfrac{m}{x} - \dfrac{n}{y}$, or $\dfrac{m+1}{x} = \dfrac{n+1}{y}$;

from (1) and (3), $\dfrac{m+1}{x} = \dfrac{p+1}{z}$; $\therefore$, substituting in (3),

$$\frac{1}{x} + \frac{m+1}{n+1} \cdot \frac{1}{x} = \frac{p}{p+1} \cdot \frac{m+1}{x}, \text{ or } \frac{1}{m+1} + \frac{1}{n+1} = \frac{p}{p+1} = 1 - \frac{1}{p+1}.$$

168. Here $a_0 + a_1x + \&c. + a_rx^r + \&c = (p_0 + p_1x + \&c + p_rx^r + \&c)^2$ $= (p_0 + p_1x + \&c + p_rx^r + \&c) \times (p_0 + p_1x + \&c + p_rx^r + \&c)$: multiply, and equate coefficients of $x^r$.

169*. (i) $\sqrt{\frac{x^2-1}{x}}+\sqrt{\frac{x-1}{x}}=\sqrt{\frac{x-1}{x}}\{\sqrt{x+1}+1\}=$ (by Question) $x=(x+1)-1$;
$\therefore \sqrt{\frac{x-1}{x}}=\sqrt{(x+1)}-1$, or $\sqrt{x}+\sqrt{x-1}=\sqrt{x+1}$; square, &c.

(ii) $\sqrt{x^2-1}\,\{1+\sqrt{x^2+1}\}=x^3=x\{(x^2+1)-1\}$; $\therefore \sqrt{x^2-1}=x\{\sqrt{x^2+1}-1\}$, or $x=x\sqrt{x^2+1}-\sqrt{x^2-1}$; whence, squaring and reducing, we get $(x^4-1)-2x\sqrt{x^4-1}+x^2=0$, or $\sqrt{x^4-1}-x=0$; $\therefore x^4-1=x^2$, &c.

170. The chance that $A$ wins at least four games out of five is [215] $p^5+5p^4q$, where $p=\frac{2}{3}$, $q=\frac{1}{3}$, and is $\therefore \frac{112}{243}$: the odds against him are, consequently, 131 : 112.

172. See Ex. 47.

173. Observe that in each case $c$ is the $(n+2)$th term of an A., G., or H. series, of which $a$ is the 1st.

176. This follows, of course, at once from [86]: but may be proved independently, as follows.

Let $B=aA$, $AC=bB^2=abAB$; then $C=abB=a^2bA$, and $\therefore C \propto B \propto A$: also let $\sqrt[3]{(A^2D)}+\sqrt[3]{(AB^2)}=cC$, or $\sqrt[3]{(A^2D)}=a^2bcA-A\sqrt[3]{a^2}$, or $D=A(a^2bc-a^{\frac{2}{3}})^3=dA$ suppose, and therefore $D \propto A$. Hence $m\sqrt[4]{(AD)}-n\sqrt[4]{(BC)} : p\sqrt{A}+q\sqrt{D}$ in the *constant* ratio $m\sqrt[4]{d}-n\sqrt[4]{(a^3b)} : p+q\sqrt{d}$.

177*. Since $(a-b)^2$ is positive, therefore $a^2>2ab-b^2>(2a-b)\,b$; so $b^2>(2b-c)\,c$, $c^2>(2c-a)\,a$;

$$\therefore a^2b^2c^2>abc\,(2a-b)\,(2b-c)\,(2c-a).$$

178. $Q_1=\frac{1}{n}A+P$, $R_1=\frac{n-1}{n}A-P$; $Q_2=\frac{1}{n}\left(\frac{n-1}{n}A-P\right)+2P$
$=Q_1=\frac{1}{n}A+P$: $\therefore A=n\,(n-1)\,P$; and then we have
$Q_1=(n-1)\,P+\ P\ =nP$, and $R_1=n\,(n-2)\,P$;
$Q_2=(n-2)\,P+2P=nP$, &c.:
$\therefore$ share of each $=nP$, and number of persons $=n-1$.

179. From (1) $(x+y)\,(x^2+y^2)=13$, and (2) ÷ (1) gives $xy=6\sqrt{(x+y)}$;
$\therefore$ by (1) $\{(x+y)^2-12\sqrt{(x+y)}\}\,(x+y)=(x+y)^3-12\,(x+y)^{\frac{3}{2}}=13$,
whence $(x+y)^{\frac{3}{2}}=-1$ or $13$;

$$\therefore x+y=1,\ \text{and}\ xy=6\sqrt{(x+y)}=-6,\ \text{\&c.}$$

180*. The possible cases are $S_2s_4$, $S_3s_3$, $S_4s_2$; $P_1=P_2=P_3$;
$p_1=\frac{2}{6}$, $p_2=\frac{3}{6}$, $p_3=\frac{4}{6}$, $\Sigma\,(p)=\frac{9}{6}$; $\therefore Q_1=\frac{2}{9}$, $Q_2=\frac{3}{9}$, $Q_3=\frac{4}{9}$,
and $B$'s exp. $=\frac{2}{9}$ of $44s+\frac{3}{9}$ of $63s+\frac{4}{9}$ of $82s=£3\ 7s\ 2\frac{2}{3}d$.

181. In [130] put $r = 10$, $a = 3$.

182. By [148] (i) $a = \frac{A}{r}(1 - R^{-n})$, (2) $a' = \frac{A}{r}(1 - R^{-2n})$;

$$\therefore \frac{a'}{a} = 1 + R^{-n}; \quad \therefore \frac{ra}{A} + \frac{a'}{a} = 2; \text{ whence } r, \text{ and } R^n = \frac{a}{a' - a}.$$

186*. Use $A$, $B$, $C$, $D$, for the four colours: in the four middle squares put $A$, $B$, $C$, $D$; this may be done in 4.3.2.1 = 24 ways: take any one of these, that, for instance, in which $AB$ is horizontal, $AC$ vertical, and $AD$ diagonal: then in the vertical line, $AC$, may be placed $B$ and $D$, and this may be done in 2 ways, the order being $BACD$ or $DACB$; so in the vertical, $BD$, may be placed $A$ and $C$, in the horizontal, $AB$, may be placed $C$ and $D$, and in the horizontal, $CD$, may be placed $A$ and $B$, each pair in 2 ways; and now it will be found, by drawing the figure in any particular case, that there is no choice as to the colours of the four remaining corner squares: hence the whole number of different ways = 24 × 2 × 2 × 2 × 2 = 384.

187. The series is $p\{a^m \pm ma^{m-1}b + \&c\} \pm mqb\{a^{m-1} \pm (m-1)a^{m-2}b + \&c\}$ $= p(a \pm b)^m \pm mqb(a \pm b)^{m-1}$.

188. Let $r_1$, $r_2$, &c. be the *ratios* of the respective series: then $y$ is the $(m+2)^{\text{th}}$ term of the series $x$, $xr_1$, $xr_1^2$, &c.;

$$\therefore y = xr_1^{m+1}, \text{ and } r_1 = \left(\frac{y}{x}\right)^{\frac{1}{m+1}}; \text{ so } r_2 = \left(\frac{2y}{x}\right)^{\frac{1}{m+1}}, \&c.;$$

$$\therefore M_1 = xr_1^m, M_2 = xr_2^m, \&c; \text{ or } \frac{M_1}{M_2} + \frac{M_2}{M_3} + \&c. = \left(\frac{r_1}{r_2}\right)^m + \left(\frac{r_2}{r_3}\right)^m + \&c.$$

$$= \left(\frac{1}{2}\right)^{\frac{m}{m+1}} + \left(\frac{2}{4}\right)^{\frac{m}{m+1}} + \&c. = m\left(\frac{1}{2}\right)^{\frac{m}{m+1}} = \sqrt[m+1]{\frac{m^{m+1}}{2^m}}.$$

189. $$\sqrt{(\sqrt[n]{a} - \sqrt[n]{x})}\left\{\frac{1}{\sqrt[n]{x}} - \frac{1}{\sqrt[n]{a}}\right\} = \frac{(\sqrt[n]{a} - \sqrt[n]{x})^{\frac{3}{2}}}{\sqrt[n]{(ax)}} = \sqrt[2n]{\frac{x}{b^2}};$$

$$\therefore \frac{(\sqrt[n]{a} - \sqrt[n]{x})^{3n}}{a^2x^2} = \frac{x}{b^2}, \text{ or } \left(\frac{\sqrt[n]{a} - \sqrt[n]{x}}{\sqrt[n]{x}}\right)^n = \frac{a^2}{b^2}, \&c.$$

190. The chance [215] $= p^4 + 4p^3q$, where $p = \frac{3}{8}$, $q = \frac{5}{8}$.

191. (i) $\sqrt[3]{m} > \sqrt[4]{(m+1)}$, if $m^4 > (m+1)^3$, or $m > \left(1 + \frac{1}{m}\right)^3 > 1 + \frac{3}{m} + \&c$, which is the case if $m > 3$.

(ii) Extract the roots as in [42 Ex. 3]: then

$$(\tfrac{1}{2} - \sqrt{-\tfrac{1}{3}}) + (\tfrac{1}{2} + \sqrt{-\tfrac{1}{3}}) = 1.$$

192. Put $\frac{a_1}{a_2} = \frac{a_2}{a_3} = \&c = m$: then

$$\frac{a_1^2 + a_2^2 + \&c.}{a_1 a_2 + a_2 a_3 + \&c.} = \frac{a_1 . m a_2 + a_2 . m a_3 + \&c.}{m a_2 . a_2 + m a_3 . a_3 + \&c.} = \&c.$$

193. (i) $\frac{(1 + \frac{1}{2}x) + (1 - \frac{2}{3}x)}{1 + x + (1 + \frac{1}{2}x)} = \frac{2 - \frac{1}{6}x}{2 + \frac{3}{2}x} = \frac{1 - \frac{1}{12}x}{1 + \frac{3}{4}x} = 1 - \frac{1}{12}x - \frac{3}{4}x = 1 - \frac{5}{6}x$:

(ii) $\frac{x^{\frac{1}{2}} + x^{\frac{2}{3}}}{x + x^{\frac{1}{3}}} = \frac{x^{\frac{2}{3}}}{x}$ nearly, since, when $x$ is very great, $x^{\frac{1}{2}}$ vanishes compared with the *higher* power, $x^{\frac{2}{3}}$, &c.

195. Assume $x . \frac{1 - bx^2 + dx^4}{1 - ax^2 + cx^4} = x(1 + Ax^2 + Bx^4 + Cx^6 + Dx^8 + \&c)$.

196. $(n^2 + 1)x^2 - 4nyx + (n^2 + 1)(y^2 - a^2) = 0$, whence

$$x = \frac{2ny \pm \sqrt{\{(n^2 + 1)^2 a^2 - (n^2 - 1)^2 y^2\}}}{n^2 + 1},$$

where we must have $y^2 < \left(\frac{n^2 + 1}{n^2 - 1}\right)^2 a^2$, that is, $y$ between $\pm \frac{n^2 + 1}{n^2 - 1} a$, which are, therefore, the limits of $y$, and also of $x$ from the symmetry of the Equation.

197. $S_n = \{2a + (n - 1) b\} \frac{1}{2}n = an + \frac{1}{2}b (n - 1) n$;
$\therefore S_n + S_{n+1} + \&c = a\{n + (n+1) + \&c\} + \frac{1}{2}b \{(n-1) n + n (n+1) + \&c\}$,
where the coefficient of $a$ is an A. P., and, in that of $\frac{1}{2}b$, the sum may be found thus:

for the series $1.2 + 2.3 + \&c$, $D = (n + 1)(n + 2)$;

$\therefore$ by [187] $S_n = \frac{1}{3}n (n + 1)(n + 2)$, and the sum required is
$S_{2n-2} - S_{n-2} = \frac{1}{3}\{(2n-2)(2n-1) 2n - (n-2)(n-1) n\} = \frac{1}{3}n (n-1)(7n-2)$.

198. Let $Ap^m + Bp^n = a$, $Aq^m + Bq^n = b$; then

$$A = \frac{aq^n - bp^n}{p^m q^n - p^n q^m},\ B = \frac{bp^m - aq^m}{p^m q^n - p^n q^m},\ \text{and } x = Ar^m + Br^n = \&c.$$

199. (2) ÷ (1) gives $(x^2 + y^2)(x^2 - xy + y^2) = 91$; $\therefore$, from (1), $(25 - 2xy)(25 - 3xy) = 91$, whence $xy$ may be found.

200. Let $x = A$'s chance: then, if $A$ fails, of which event the chance is $\frac{3}{4}$, $B$'s chance will be $x$, and $\therefore$ $B$'s chance at first is $\frac{3}{4}x$; so $C$'s is $\frac{9}{16}x$, $D$'s $\frac{27}{64}x$: and the sum of all these chances is certainty; or $x + \frac{3}{4}x + \frac{9}{16}x + \frac{27}{64}x = 1$.

201. $S' = \{1 + n + \frac{n(n-1)}{1.2} + \&c\} + \{n + n(n-1) + \frac{n(n-1)(n-2)}{1.2.3} + \&c\}$

$= (1+1)^n + n(1+1)^{n-1} = 2^n + n.2^{n-1} = (n+2)\,2^{n-1}$.

Similarly, with the lower signs, $S = (1-1)^n + n(1-1)^{n-1} = 0$.

204. By [148] the fine $P$ may be regarded as the present value of an annuity, or *extra-rent*, $P.\frac{R-1}{1-R^{-n}} = A$, suppose: the new fine then must provide for this extra-rent during the $q-p$ years, which are to be added to the original term = fine for $q$ years − fine for $p$ years $= \frac{A}{R-1}\{(1-R^{-q}) - (1-R^{-p})\} = \&c.$

205. By [219 Ex. 1] $B$'s chance of throwing 6 is $\frac{5}{36}$, and of throwing above 6 is $\frac{21}{36}$; ∴ his expectation is

$$(\tfrac{5}{36} \text{ of } 1s + \tfrac{21}{36} \text{ of } 2s) - 1s = \tfrac{11}{36}s = 3\tfrac{2}{3}d.$$

206. (ii) $a^p(1+h)^{-m} + (b^q - a^p)(1+h)^{-n} = a^p(1-mh) + (b^q - a^p)(1-nh)$

$= b^q + h\{(n-m)a^p - nb^q\} = b^q[1 + \{(n-m)a^p b^{-q} - n\}h]$

$= b^q(1+kh)$ suppose $= b^q(1+h)^k = \&c.$

207. $$\frac{a^3 - x^3}{(x^3 - y^3) - (a^3 - b^3)} = \frac{a^3 - x^3}{(x^3 - y^3) - (x-y)^3} = \frac{a^3 - x^3}{3xy(x-y)} = (85)\ \frac{2x+3y}{x-y};$$

$\therefore\ a^3 - x^3 = 3xy(2x+3y)$, and, by (1), $b^3 - y^3 = 3xy(3x+2y)$;

hence $a^3 = x^3 + 6x^2y + 9xy^2 = (x^{\frac{3}{2}} + 3x^{\frac{1}{2}}y)^2$, or $a^{\frac{3}{2}} = x^{\frac{3}{2}} + 3x^{\frac{1}{2}}y$;

$$\text{so } b^{\frac{3}{2}} = y^{\frac{3}{2}} + 3xy^{\frac{1}{2}};\ \text{ and }\ a^{\frac{3}{2}} + b^{\frac{3}{2}} = (x^{\frac{1}{2}} + y^{\frac{1}{2}})^3 = z^{\frac{3}{2}}.$$

208. $S_1 = 1 + 1 + \&c = n,\ S_2 = \frac{2^n - 1}{1},\ S_3 = \frac{3^n - 1}{2},\ \&c,\ S_n = \frac{n^n - 1}{n-1}.$

209. Let $S = 1 + \frac{1}{2} + \frac{1}{3} + \&c + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}$,

$\therefore S - (1 + \frac{1}{2} + \frac{1}{3}) = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c + \frac{1}{n+3}$; subtracting, we get

$$1 + \tfrac{1}{2} + \tfrac{1}{3} = \frac{3}{1.4} + \frac{3}{2.5} + \frac{3}{3.6} + \&c + \frac{3}{n(n+3)} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3};$$

$$\text{or } 3S = (1 - \frac{1}{n+1}) + (\frac{1}{2} - \frac{1}{n+2}) + (\frac{1}{3} - \frac{1}{n+3}) = \&c.$$

211. $\frac{x^{-m}}{x^{-m} + x^{-n} + x^{-p}} + \&c.$ 212. $\sqrt[m]{a} - \sqrt[m]{(a-x)} : \sqrt[n]{(b+x)} - \sqrt[n]{b}$

$= \sqrt[m]{a} - \sqrt[m]{a}\,(1 - \frac{1}{m}\frac{x}{a}) : \sqrt[n]{b}\,(1 + \frac{1}{n}\frac{x}{b}) - \sqrt[n]{b} = nb\sqrt[m]{a} : ma\sqrt[n]{b} = \&c.$

215. (i) There are 36 different throws with two dice: the ace of *either* die may fall with any one of the five other numbers of the other die, so that there are $2 \times 5 = 10$ favorable throws; and the chance of a single ace is $\frac{10}{36}$:

(ii) One ace must be thrown the first time, and none the next, or *vice versa;* now there are 216 different throws with three dice, and the ace of either of the three may fall with $5 \times 5 = 25$ different pairs of numbers on the other two dice;

$\therefore$ chance of *one* ace, in either throw, $= \dfrac{3 \times 25}{216} = \dfrac{25}{72}$, and of

no ace $= \dfrac{5 \times 5 \times 5}{216} = \dfrac{125}{216}$; hence chance required $= 2\left(\dfrac{25}{72} \times \dfrac{125}{216}\right)$:

and the ratio of the two chances in (i) and (ii) is 432 : 625.

216. Put $b^2 - 4ac = (b - 2am)^2$, &c.

217. Let $x =$ n° of candidates; then, voting only for *one*, he can dispose of his vote in $x$ ways; voting for *two*, in $\dfrac{x(x-1)}{1.2}$ ways, &c; $\therefore x + \dfrac{x(x-1)}{1.2} + \&c + x = n = 2^x - 2$, by (177 Cor).

218. Let $\dfrac{10r_m}{b} = q_{m+1} + \dfrac{r_{m+1}}{b}$, $\dfrac{10r_n}{b} = q_{n+1} + \dfrac{r_{n+1}}{b}$, and $r_m + r_n = b$;

$\therefore 10 = q_{m+1} + q_{n+1} + \dfrac{r_{m+1} + r_{n+1}}{b}$; hence $r_{m+1} + r_{n+1}$, must be a multiple of $b$, and (each of $r_{m+1}$ and $r_{n+1}$ being $< b$) must therefore $= b$, and then $q_{m+1} + q_{n+1} = 9$.

219. (i) $= \dfrac{1}{6}\left\{\dfrac{1}{1.3} - \dfrac{1}{2.4} + \dfrac{1}{3.5} - \&c\right\} = \frac{1}{6}S$ suppose.

$$\text{Now let } S' = x + \frac{x^2}{2} + \frac{x^3}{3} + \&c + \frac{x^n}{n} + \frac{x^{n+1}}{n+1} + \frac{x^{n+2}}{n+2};$$

$$\text{then } \frac{S'}{x^2} = \frac{x}{3} + \frac{x^2}{4} + \frac{x^3}{5} + \&c + \frac{x^n}{n+2} + \frac{1}{x} + \frac{1}{2};$$

$$\therefore S'\left(1 - \frac{1}{x^2}\right) = \frac{2x}{1.3} + \frac{2x^2}{2.4} + \frac{2x^3}{3.5} + \&c + \frac{2x^n}{n(n+2)} + \frac{x^{n+1}}{n+1} + \frac{x^{n+2}}{n+2} - \frac{1}{x} - \frac{1}{2};$$

put $x = -1$; then

$$0 = -\frac{2}{1.3} + \frac{2}{2.4} - \frac{2}{3.5} + \&c \pm \frac{2}{n(n+2)} + \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+2}}{n+2} + 1 - \frac{1}{2},$$

$$\text{or } 2S = \frac{1}{2} - (-1)^n\left\{\frac{1}{n+1} - \frac{1}{n+2}\right\} = \frac{1}{2} - \frac{(-1)^n}{(n+1)(n+2)}.$$

(ii) $= \frac{1}{3}(\frac{1}{1.2} - \frac{1}{2.4}) + \frac{1}{3}(\frac{1}{2.3} - \frac{1}{3.5}) + \&c = \frac{1}{3}(\frac{1}{1.2} + \frac{1}{2.3} + \&c)$
$- \frac{1}{3}(\frac{1}{2.4} + \frac{1}{3.5} + \&c) = \frac{1}{3}\{(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \&c + (\frac{1}{n} - \frac{1}{n+1})\}$
$- \frac{1}{6}\{(\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + \&c + (\frac{1}{n} - \frac{1}{n+2}) + (\frac{1}{n+1} - \frac{1}{n+3})\}$
$= \frac{1}{3}(1 - \frac{1}{n+1}) - \frac{1}{6}\{(\frac{1}{2} - \frac{1}{n+2}) + (\frac{1}{3} - \frac{1}{n+3})\} = \&c.$

220. $B$'s exp. arises from the sum of the different chances of $A$'s throwing (i) *head*, (ii) *tail-head*, (iii) *tail-tail-head*, and

$$\therefore = \tfrac{1}{2}(1s) + \tfrac{1}{4}(2s) + \tfrac{1}{8}(3s) + \&c.$$

the sum of which series may be found by [184].

221. $\frac{x}{y} = \left(\frac{a}{b}\right)^{\frac{m}{m+n}}$, $\frac{x}{z} = \left(\frac{a}{c}\right)^{\frac{m}{m+n}}$; write $P$ for $a^{\frac{mn}{m+n}} + b^{\frac{mn}{m+n}} + c^{\frac{mn}{m+n}}$: then
$a^m + b^m . \frac{x^m}{y^m} + c^m . \frac{x^m}{z^m} = x^m = a^m \left\{1 + \left(\frac{b}{a}\right)^{\frac{mn}{m+n}} + \left(\frac{c}{a}\right)^{\frac{mn}{m+n}}\right\}$, or $\frac{a^m}{x^m} = \frac{a^{\frac{mn}{m+n}}}{P}$;

hence $x^m = a^{\frac{m^2}{m+n}}P$, and $x^n = a^{\frac{mn}{m+n}}P^{\frac{n}{m}}$; so $y^n = \&c$, $z^n = \&c$;

and $\therefore x^n + y^n + z^n = P.P^{\frac{n}{m}} = P^{\frac{m+n}{m}} = k^n$; or $P = k^{\frac{mn}{m+n}}$.

223. (i) $\{2a + (n-1)d\}\frac{1}{2}n = n^2$, or $2a - d = n(2-d)$, which is true for all values of $n$ only by having $2a = d$, $2 = d$; $\therefore a = 1$, and the series is $1 + 3 + 5 + \&c.$

(ii)* $\frac{S_2}{S_1} = r^n + 1$, $\frac{S_3}{S_1} = r^{2n} + r^n + 1 = (r^n + 1)^2 - r^n = \frac{S_2^2}{S_1^2} - (\frac{S_2}{S_1} - 1)$.

224. $\frac{\sqrt[n]{x}}{m\sqrt[n]{y}} = \frac{\sqrt[2n]{x}}{m\sqrt[2n]{(x-y)}}$; $\therefore \frac{x^2}{y^2} = \frac{x}{x-y}$, or $x^2 - xy = y^2$, and
$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0$, &c.

225. The chance of safe return for each ship $= \frac{9}{10} = p$; the chance required $= p^5 + 5p^4q + 10p^3q^2$.

226. Let $x^2 + y^2 + z^2 = a^2$, and put $y^2 = 2xz$; then $x + z = a$, or $x = a - z$; $\therefore 2(a-z)z = \square = m^2z^2$ suppose; and

$$z = \frac{2a}{m^2+2}, \quad x = \frac{m^2a}{m^2+2}, \quad y = \frac{2ma}{m^2+2}.$$

If $a = 9$, put $m = 4$; then $x = 8$, $y = 4$, $z = 1$.

227. Let $x =$ n° of candidates, which is odd; then $\frac{1}{2}(x-1) =$ n° to be elected; and $n = x + \frac{x(x-1)}{1.2} + \&c + \frac{x(x-1)\ldots\{x-\frac{1}{2}(x-1)+1\}}{1.2\ldots\frac{1}{2}(x-1)}$ $= S$, suppose: but $(1+1)^x = 2 + 2S$; $\therefore S = 2^{x-1} - 1 = n$, and $x = 1 + \frac{\log(n+1)}{\log 2} = \frac{\log\{2(n+1)\}}{\log 2}$.

228. By [95 Ex. 5] $A_m = \frac{1.3\ldots(2m-1)}{1.2\ldots m}(2x)^m$;

$$\therefore A_0 + A_1 + \&c = 1 + 2x + \frac{1.3}{1.2}(2x)^2 + \&c = \{1 - 2(2x)\}^{-\frac{1}{2}}.$$

229. (i) $(n+1)S = (n+1) + \frac{(n+1)n}{1.2} + \frac{(n+1)n(n-1)}{1.2.3} + \&c.$
$= (1+1)^{n+1} - 1 = 2^{n+1} - 1.$

(ii) By [188], $D = (n+1)^{\text{th}}$ term $= (n+1)(n+2)^2$
$= (n+1)(n+2)(n+3-1) = (n+1)(n+2)(n+3) - (n+1)(n+2)$;
$\therefore S = \frac{1}{4}n(n+1)(n+2)(n+3) - \frac{1}{3}n(n+1)(n+2) = \&c.$

230. Let the number be $n$; then n° of ways in which they can be drawn 1, 3, 5, &c. together $= n + \frac{n(n-1)(n-2)}{1.2.3} + \&c = 2^{n-1}$ by [97 Ex. 6]; and n° of ways in which they can be drawn 2, 4, 6, &c. together

$$= \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} + \&c = 2^{n-1} - 1.$$

232. Let $a, b, c,$ be the roots; then [58]

$$-p = a + b + c, \quad q = ab + ac + bc, \quad -r = abc;$$

therefore, if $r = pq$, we have
$abc = (a+b+c)\{a(b+c)+bc\} = a^2(b+c) + a(b+c)^2 + bc(b+c) + abc$;
hence either $b + c = 0$, (which proves at once the statement,)
or else $\quad a^2 + a(b+c) + bc = 0 = (a+b)(a+c)$,
so that either $a + b = 0$, or $a + c = 0$.

233. Since $315 = 5.7.9$, assume

$$\frac{x}{5} + \frac{y}{7} + \frac{z}{9} = \frac{401}{315}, \quad \text{or } 63x + 45y + 35z = 401;$$

$$\therefore 45y + 35z = 401 - 63x,$$

where the second side must be a multiple of 5, since the first side is, and $\therefore x = 2$; then $9y + 7z = 55$, where $y = 3$, $z = 4$.

235. Here $p : q :: m : n$: now in $(p+q)^x$, the $y^{\text{th}}$ term is the greatest [94], when $y$ has the least value which exceeds $(x+1)\dfrac{q}{p+q} =$ (in this case) $\{r(m+n)+1\}\dfrac{n}{m+n} = rn + \dfrac{n}{m+n}$; hence the $(rn+1)^{\text{th}}$ term is the greatest, and this involves the factor $p^{rm}q^{rn}$, which indicates the most probable event.

236. $X - a'' = ax + a'x'$, $Y - b'' = bx + b'x'$, $Z - c'' = cx + c'x'$: hence

$$(X-a'')(bc'-b'c) + (Y-b'')(ca'-ac') + (Z-c'')(ab'-ba') = 0,$$

or $l(X-a'') + m(Y-b'') + n(Z-c'') = 0$, suppose, whence

$$lX + My + nZ = la'' + mb'' + nc'' = P, \text{ suppose}:$$

but from (1) and (2) $lY = mX$; $\therefore \dfrac{X}{l} = \dfrac{Y}{m} = \dfrac{Z}{n}$, by Symmetry;

and $X\left(l + \dfrac{m^2}{l} + \dfrac{n^2}{l}\right) = P$, or $X = \dfrac{lP}{l^2+m^2+n^2}$, $Y =$ &c, $Z =$ &c,

and $X^2 + Y^2 + Z^2 = \dfrac{P^2}{l^2+m^2+n^2}$.

237. In 1st operation, wine drawn $= b$, and wine remaining $= a - b$;

in 2nd, wine drawn : $b :: a - b : a$, and $\therefore = (a-b)\dfrac{b}{a}$,

and wine remaining $= (a-b)\left(1 - \dfrac{b}{a}\right) = \dfrac{(a-b)^2}{a}$;

in 3rd, wine drawn : $b :: \dfrac{(a-b)^2}{a} : a$, and $\therefore = \dfrac{(a-b)^2}{a}\,\dfrac{b}{a}$,

and wine remaining $= \dfrac{(a-b)^2}{a}\left(1 - \dfrac{b}{a}\right) = \dfrac{(a-b)^3}{a}$; &c.

238. $x = \{1 - (1 - x^{-\frac{n}{m}})\}^{-\frac{m}{n}}$; $\therefore \log_e x = -\dfrac{m}{n}\log_e\{1 - (1 - \sqrt[m]{x^{-n}})\} =$ &c.

239. $S$ ($n$ odd) $= 1 + 3 +$ &c. to $\frac{1}{2}(n+1)$ terms $+$ $4\{1^2 + 2^2 +$ &c. to $\frac{1}{2}(n-1)$ terms$\} = \frac{1}{4}(n+1)^2 + \frac{1}{6}n(n^2-1) =$ &c: similarly, when $n$ is *even*.

240. The chance that $A$ will throw tails three times successively is $\frac{1}{8}$; $\therefore$ $B$'s exp. is $\frac{1}{8}$ of $10s - \frac{7}{8}$ of $1s = 4\frac{1}{2}d$; and the bet should have been as $\frac{7}{8} : \frac{1}{8}$, or $7s$ to $1s$.

242. Put $x = 1 + h$; then (i) $m(1+mh) - n(1+nh)$

$$= (m-n)\{1 + (m+n)h\} = (m-n)(1+h)^{m+n} \text{ nearly.}$$

(ii) $(1+h)^{-e} = 1 - eh = (1-2eh)^{\frac{1}{2}} = \{1 - e(1+h) + e(1-h)\}^{\frac{1}{2}}$

$= \{1 - e.e^h + e.e^{-h}\}^{\frac{1}{2}}$ nearly $= \{1 - e^{1+h} + e^{1-h}\}^{\frac{1}{2}} = \sqrt{(1 - e^x + e^{\frac{1}{x}})}$.

244. $S_{2n} = \{2 + (2n-1)b\}n$, $S_n = \{2 + (n-1)b\}\frac{1}{2}n$;
let $S_n : S_{2n} - S_n :: m : 1 = 2 + (n-1)b : 2 + (3n-1)b$;
$= (2-b) + nb : (2-b) + 3nb$;
whence $(m-1)(2-b) = nb(1-3m)$ for all values of $n$;
$\therefore$ [170] $1 - 3m = 0$ or $m = \frac{1}{3}$, and $2 - b = 0$ or $b = 2$.

245. The question is the same as if there were to be *one* trial (i) with six dice, (ii) with twelve: in (i) the aces might be on any two of the dice, (and the six may be taken, two together, in 15 ways,) and on the other four may be any of the five other numbers, so that the whole n° of *favorable* throws is $15 \times 5^4$, and the whole n° of *possible* throws is $6^6$;
$\therefore$ chance in (i) $= \dfrac{15.5^4}{6^6}$: and similarly chance in (ii) $= \dfrac{66.5^{10}}{6^{12}}$.

246. $2x_1 \times (2x_1 + 2x_2 + \&c) = (y^z - y^{-z})\{(y^z - y^{-z}) + (y^{3z} - y^{-3z}) + \&c\}$
$= (y^z - y^{-z})\{(y^z + y^{3z} + \&c) - (y^{-z} + y^{-3z} + \&c)\}$
$= (y^z - y^{-z})\left\{y^z \cdot \dfrac{y^{(2n-1)2z} - 1}{y^{2z} - 1} - y^{-z} \cdot \dfrac{y^{(1-2n)2z} - 1}{y^{-2z} - 1}\right\}$
$= (y^{(2n-1)2z} - 1) - (y^{(1-2n)2z} - 1) = (y^{(2n-1)z} - y^{(1-2n)z})^2 = 4x_n^2$.

247. Assume $x^4 + px^3 + qx^2 + rx - s^2 = (x^2 + mx + s)(x^2 + nx - s) = \&c$;
$\therefore\ p = m + n$, $q = mn$, $\dfrac{r}{s} = n - m$, whence $2n = p + \dfrac{r}{s}$,
$2m = p - \dfrac{r}{s}$, $4mn = p^2 - \dfrac{r^2}{s^2} = 4q$.

Ex. $2m = -6 - 4 = -10$, $2n = -6 + 4 = -2$;
$\therefore\ x^3 - 5x + 2 = 0$, and $x^2 - x - 2 = 0$.

248. $z = y(1 - y^2)^{-\frac{1}{2}} = y + \frac{1}{2}y^3 + \frac{3}{8}y^5 + \&c$; and since $y$ changes *sign*, not *value*, with $z$, and $z$ with $x$, we may assume

$$y = ax + bx^3 + cx^5 + \&c;$$

$$\begin{aligned}\therefore\ z &= (ax + bx^3 + cx^5 + \&c) + \tfrac{1}{2}(a^3x^3 + 3a^2bx^5 + \&c) + \tfrac{3}{8}a^5x^5 + \&c.\\ &= ax + (b + \tfrac{1}{2}a^3)x^3 + (c + \tfrac{3}{2}a^2b + \tfrac{3}{8}a^5)x^5 + \&c:\end{aligned}$$

$$\begin{aligned}\therefore\ x = z - \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 - \&c = ax &+ (b + \tfrac{1}{2}a^3)x^3 + (c + \tfrac{3}{2}a^2b + \tfrac{3}{8}a^5)x^5 + \&c.\\ &- \tfrac{1}{3}a^3\ \ x^3 - \qquad (a^2b + \tfrac{1}{2}a^5)x^5 - \&c.\\ &\qquad\qquad\qquad\qquad + \tfrac{1}{5}a^5\ \ x^5 + \&c.\end{aligned}$$

$\therefore\ 1 = a$, $0 = b + \frac{1}{6}a^3$, or $b = -\frac{1}{6}$; $0 = c + \frac{1}{2}a^2b + \frac{3}{40}a^5$, or $c = \frac{1}{120}$.

249. See [74]. 250. See [217 Ex. 2]: *B*'s drawing, whether before or after *A*'s, if his coin be not looked at, makes no difference at all in *A*'s expectation; just as if *A* had had originally *three* chances, and had given *B* one of them.

251. $\frac{a-b-d+c}{a-b-c+d}=\frac{ac-bd}{ad-bc}$; $\therefore \frac{a-b}{c-d}=\frac{(ac-bd)+(ad-bc)}{(ac-bd)-(ad-bc)}=\frac{(a-b)(c+d)}{(a+b)(c-d)}$;

$$\therefore a+b=c+d, \text{ or } a-c=d-b; \text{ hence}$$

$$\frac{ad-bc}{a-b-c+d}=\frac{a(d-b)+b(a-c)}{2(a-c)}=\frac{a+b}{2}=\frac{a+b+c+d}{4}.$$

252. (i) Here $b^2=ac$; and $a^2-b^2+c^2>(a-b+c)^2$, if (expanding and arranging) $2ab+2bc>2ac+2b^2>4ac$, or $b>\frac{2ac}{a+c}$, which it is, by [150].

(ii) $(a-a')^2+(b-b')^2+(c-c')^2$ is *positive*, and $\therefore >0$; that is $1+1-2(aa'+bb'+cc')>0$.

255. It matters not *when* $B$ drew his coin; it is certain that $A$ cannot have drawn $B$'s sovereign; and $\therefore$ his expectation is the same as if he had to draw two coins out of *eight* only, which he knows to be four shillings and four sovereigns, that is, it is $\frac{2}{8}$ of 84*s*.

256. Squaring and multiplying out, it will be found that the quantity may be put into the form

$$\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right)^2+\left(\frac{yx'-xy'}{ab}\right)^2=0.$$

257. Assume for the numbers $4x$, $x^2-4x$, and $2x+1$: then three of the conditions are satisfied, and it remains to make the sum of the *first* and *third* a square: put $6x+1=m^2$; $\therefore x=\frac{1}{6}(m^2-1)$: put $m=11$.

258. $\frac{b+1}{b-1}=\frac{a(1+x+x^2+\&c)}{xy(1+2x+3x^2+\&c)}=\frac{a(1-x)^{-1}}{xy(1-x)^{-2}}=\frac{a(1-x)}{xy}$;

$\therefore y=a\frac{b-1}{b+1}(\frac{1}{x}-1)$, and if $x$ increase in H. P., then $\frac{1}{x}$, and $\therefore$ also $y$, will increase in A. P.

259. (i) $\frac{(n+1)^{\text{th}}}{n^{\text{th}}}=\frac{x^2}{2n(2n+1)}=\frac{x^2}{4n^2+2n}$, which is $<1$, if $4n^2+2n>x^2$, or $2n+\frac{1}{2}>\sqrt{(x^2+\frac{1}{4})}$; $\therefore n$ must be the integer next $>\frac{1}{2}\{\sqrt{(x^2+\frac{1}{4})}-\frac{1}{2}\}$, that is, $n=\frac{1}{2}x$.

(ii) $(x-x^{-1})^n=x^n(1-x^{-2})^n$: find the greatest term of $(1-x^{-2})^n$ by [94].

260. Their chances are equal, and each $=\frac{3}{7}$: (ii) $\frac{3}{7}\times\frac{4}{7}\times\frac{3}{7}=\frac{36}{343}$: (iii) $\frac{3}{7}\times\frac{3}{7}\times\frac{3}{7}=\frac{27}{343}$.

261. Let $x = \dfrac{12ab}{a+b+p}$ suppose; then

$$\frac{x+2a}{x-2a} = \frac{12ab+2a(a+b+p)}{12ab-2a(a+b+p)} = \frac{(7b+a)+p}{(5b-a)-p} = \frac{(7b+a+p)(5b-a+p)}{(5b-a)^2-p^2}$$

$$= \frac{35b^2-2ab-a^2+12bp+p^2}{25b^2-10ab+a^2-p^2} = \text{(substituting for } p^2\text{)}\ \frac{3b+a+p}{2(b-a)}:$$

$$\text{so } \frac{x+2b}{x-2b} = \frac{3a+b+p}{2(a-b)}, \text{ by Symmetry;}$$

and their sum $= 1$.

262. Put $2x^2 - \&c = (2x+ly+m)(x+ny+p)$; then, equating coeff^s, $l+2n=-21$, $ln=-11$, $mn+lp=34$, $m+2p=-1$, $mp=-3$; from the first two of these we see that $l$ and $2n$ are the roots of $z^2+21z-22=0$, where $z=1$ or $-22$; from the last two, $m$ and $2p$ are the roots of $z^2+z-6=0$, where $z=2$ or $-3$; if we take $m=-3$ and $2p=2$, we have, from the middle equation, $-3n+l=34$, where we see that, of the two roots (1 and $-22$), $2n$ must $=-22$ and $l=1$, and the factors are, consequently, $(2x+y-3)(x-11y+1)$.

263. Let $\dfrac{n^2-n+1}{n^2+n+1} = x$: then $(1-x)n^2-(1+x)n+(1-x)=0$,

and $n = \dfrac{1+x \pm \sqrt{\{(1+x)^2-4(1-x)^2\}}}{2(1-x)}$, which requires that

$\dfrac{1+x}{1-x}$ should lie between $+2$ and $-2$, or $x$ between $\frac{1}{3}$ and 3.

264. For shortness, write $f$ and $f'$ for $f(x)$ and $f'(x)$: then $(a^x-f)^2 = f^2-1$, whence $f=\frac{1}{2}(a^x+a^{-x})$, $f'=\frac{1}{2}(a^x-a^{-x})$;

$$\therefore f(x \pm y) = \tfrac{1}{2}(a^{x\pm y}+a^{-(x\pm y)})$$
$$= \tfrac{1}{4}\{(a^x+a^{-x})(a^y+a^{-y}) \pm (a^x-a^{-x})(a^y-a^{-y})\} = \&c.$$

265. (i) $\frac{3}{12}$ of £5 9*s*: (ii) $\frac{3}{12}$ of £4 8*s*.

266. The $x$th term is $\frac{25}{55}\{12.10^{x-2}-(-1)^{x-2}\}$, which is integral, if $11.10^{x-2}+10^{x-2}-(-1)^{x-2}$ is divisible by 11: but $10^{x-2} \pm 1$ is divisible by $10+1$, or 11, according as $x$ is odd or even, &c.

Since the $x$th term may be written $\frac{6}{11} \cdot 10^{x-1} + \frac{5}{11} \cdot (-1)^{x-1}$,

we have $S = \dfrac{6}{11} \cdot \dfrac{10^x-1}{10-1} + \dfrac{5}{11} \cdot \dfrac{(-1)^x-1}{-1-1} = \&c.$

267. $N = n^2-1 = 10m^2$, whence $n^2-10m^2=1$: find $m$ and $n$ as in [166], and thence find $N$.

268. Here $5x^2 > 500 \,.\, 7x^3 (1 + \frac{2}{7}x + \frac{2}{7} \,.\, \frac{11}{9}x^2 + \&c)$

$$> 500 \,.\, 7x^3 (1 + \tfrac{2}{7}x + \tfrac{2}{7} \,.\, \tfrac{2}{7}x^2 + \&c),$$

$$\text{or } 1 > \frac{700x}{1 - \frac{2}{7}x}, \text{ whence } x < \tfrac{7}{4909} < \tfrac{1}{702} \text{ nearly.}$$

269. Let $Q_r$ = share of $r^{\text{th}}$ person $= rP + \frac{1}{n}$ th of rem$^{\text{r}}$ $(R)$;

$$\therefore R = n(Q_r - rP), \text{ and } Q_{r+1} = (r+1)\,P + \frac{1}{n}\{\frac{n-1}{n}\,R - (r+1)\,P$$

$$= \frac{n-1}{n}\,Q_r + \frac{n-1}{n}\,P = vQ_r + vP, \text{ writing } v \text{ for } \frac{n-1}{n}:$$

$$\text{hence } nP = Q_n = vQ_{n-1} + vP = (v^2Q_{n-2} + v^2P) + vP$$

$$= (v^3Q_{n-3} + v^3P) + v^2P + vP = \&c.$$

$$= (v^{n-1}Q_1 + v^{n-1}P) + v^{n-2}P + \&c + vP$$

$$= v^{n-1}\,(\frac{x}{n} + vP) + v^{n-1}P + \&c + vP$$

$$= \frac{v^n x}{n-1} + vP\,\frac{1-v^n}{1-v} = \frac{v^n x}{n-1} + (n-1)\,P\,(1-v^n);$$

whence $x = (n-1)\,P\,\{(n-1) + v^{-n}\} = \&c.$

270. Four cases are, *a priori*, possible, and chance of drawing from either bag is $\frac{1}{2}$; (i) $S$ may be transferred from $A$, for which $P_1 = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$, or (ii) $s$ from $A$, for which $P_2 = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$, or (iii) $S$ from $B$, for which $P_3 = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$, or (iv) $s$ from $B$, for which $P_4 = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$.

In (i) $A$ contains now $SSs$, $B$ $SSsss$; $\therefore p_1 = \frac{1}{3} \times \frac{3}{5} = \frac{1}{5}$, $P_1p_1 = \frac{3}{40}$;
in (ii) $A$ . . . . $SSS$, $B$ $Sssss$; $\therefore p_2 = 0 \times \frac{4}{5} = 0$, $P_2p_2 = 0$;
in (iii) $A$ . . . . $SSSSs$, $B$ $sss$; $\therefore p_3 = \frac{1}{5} \times 1 = \frac{1}{5}$, $P_3p_3 = \frac{1}{40}$;
in (iv) $A$ . . . . $SSSss$, $B$ $Sss$; $\therefore p_4 = \frac{2}{5} \times \frac{2}{3} = \frac{4}{15}$, $P_4p_4 = \frac{4}{40}$:

$$\therefore\ Q_1 = \tfrac{3}{8},\ Q_2 = 0,\ Q_3 = \tfrac{1}{8},\ Q_4 = \tfrac{4}{8} = \tfrac{1}{2}.$$

Now a second similar drawing is only possible in the case corresponding to $Q_4$: we have then $\frac{1}{2}$ the chance that there was in $A$ $SSSss$, in $B$ $Sss$; $\therefore \frac{1}{2}(\frac{2}{5} \times \frac{2}{3}) = \frac{2}{15}$ = chance that $s$ will be drawn from each in the *first* drawing, when there would remain in $A$ $SSSs$, in $B$ $Ss$;

$$\therefore\ \tfrac{2}{15}\,(\tfrac{1}{4} \times \tfrac{1}{2}) = \tfrac{1}{60} = \text{chance required.}$$

271. Multiply numerator and denominator in (1) by 1.2.3 ... $n$ then the expression becomes $\dfrac{\lfloor 2n+1}{\lfloor n} = \&c.$ as in [97 Ex. 5].

272. $\frac{a}{bx}=\frac{b}{cx}$; $\therefore$ $ac=b^2$, and $a, b, c$ are in G.P.: also

$$x\log a=y\log b=z\log c;$$

$$\therefore \frac{1}{x}+\frac{1}{z}=\frac{1}{y}\left\{\frac{\log a+\log c}{\log b}\right\}=\frac{1}{y}\cdot\frac{2\log b}{\log b}=\frac{2}{y};$$

$\therefore$ $x, y, z$ are in H.P.

273. That is, in $\{1+x(1+2x+3x^2+\&c)\}^2=\{1+x(1-x)^{-2}\}^2$

$$=\left\{1+\frac{x}{(1-x)^2}\right\}^2=1+2x(1-x)^{-2}+x^2(1-x)^{-4},$$

where the coefficient of $x^n$ in $2x(1-x)^{-2}$ will be $2n$ and in $x^2(1-x)^{-4}$ will be $\frac{1}{6}n(n^2-1)$, by the Binomial Theorem.

274. Perform on the two given quantities the operation for finding the G.C.M.; then the last remainder will be $y-5$: put this $=0$, or $y=5$; then (64) the G.C.M. of the two quantities will be the last divisor $yx+4$ or $5x+4$.

275. It may be (i) white, (ii) red, (iii) of some colour different from either; $P_1=\frac{1}{3}=P_2=P_3$: in (i), there will be $n+1$ *white*, $n$ *red;*

$$\therefore p_1=\frac{n+1}{2n+1}\cdot\frac{n}{2n}\cdot\frac{n}{2n-1}\cdot\frac{n-1}{2n-2}\cdots\frac{2}{3}\cdot\frac{1}{2}=\frac{\lfloor\underline{n+1}}{2^n.1.3.5\ldots(2n+1)}:$$

in (ii), there will be $n+1$ *red*, $n$ *white;* and $p_2$ is found as before, and $=p_1$; in (iii), there will be $n$ *red*, $n$ *white*, and one of another colour; $\therefore p_3=\frac{\lfloor\underline{n}}{2^n.1.3.5\ldots(2n+1)}$:

hence $Q_1=Q_2=\frac{\lfloor\underline{n+1}}{2\lfloor\underline{n+1}+\lfloor\underline{n}}=\frac{n+1}{2n+3}$; $Q_3=\frac{1}{2n+3}$.

276. By Trigonometry, area $=\sqrt{\{S(S-a)(S-b)(S-c)\}}$, where $S=\frac{1}{2}(a+b+c)$, and $a, b, c$, are the sides: if $m$ be the common ratio, so that $a=mgh(k^2+l^2)$, &c., it will be found easily that $S=mhl(gl+hk)$, $S-a=mhk(hl-gk)$, $S-b=mgl(hl-gk)$, $S-c=mgk(gl+hk)$; and therefore area $=m^2ghkl(gl+hk)(hl-gk)$.

277. In (i), $S=a^{m+1}\cdot\frac{a^{p-1}-1}{a-1}$, where the sum must plainly be integral: but [197] $a^{p-1}-1$ is of the form $Mp$, and $a-1$ is not a multiple of $p$; $\therefore$ $S$ is.

In (ii), $S = a^{m+1} \cdot \dfrac{a^{m(p-1)} - 1}{a - 1}$: now since $a$ is prime to $p$, so also is $a^m$, and $\therefore (a^m)^{p-1} - 1$ is of the form $Mp$, &c.

278. $$S = \frac{2}{3.5} - \frac{3}{5.7} + \&c - (-1)^n \frac{n+1}{(2n+1)(2n+3)};$$

$$\therefore S - \frac{2}{15} = -\frac{3}{5.7} + \frac{4}{7.9} - \&c.$$

and $2S - \dfrac{2}{15} = \{\dfrac{1}{3.7} - \dfrac{1}{5.9} + \&c \text{ to } (n-1) \text{ terms}\} - (-1)^n \dfrac{n+1}{(2n+1)(2n+3)}$:

now $\{\dfrac{1}{3.7} - \&c\} = \frac{1}{4}\{(\frac{1}{3} - \frac{1}{7}) - (\frac{1}{5} - \frac{1}{9}) + (\frac{1}{7} - \frac{1}{11}) - \&c\}$

$$= \tfrac{1}{4}\{\tfrac{1}{3} - \tfrac{1}{5} + (-1)^n (\frac{1}{2n+1} - \frac{1}{2n+3})\}$$

$$= \frac{1}{2}\{\frac{1}{15} + \frac{(-1)^n}{(2n+1)(2n+3)}\}; \therefore \&c.$$

279. Let $\alpha, \beta$, be the roots, $\alpha$ much $> \beta$: then $p = \alpha + \beta$, $q = \alpha\beta$, and $\alpha = p - \beta = p$ for 1st App., $= p - \dfrac{q}{\alpha} = p - \dfrac{q}{p}$ for 2nd App. $= p - \dfrac{q}{p - \dfrac{q}{p}} = \&c$, for 3rd App.

280. As in Ex. 97, the *value* of all the arrangements that can be made with the five digits, 1, 2, 3, 4, 5, in the scale of 10, is $\lfloor 4\,(1+2+3+4+5)\dfrac{10^5 - 1}{10 - 1} = \lfloor 5 \times 33333$: hence, in this case, the value of the different sums that can be made is $\lfloor 5 \times 33333$ farthings; and the number of different arrangements that can be made with five digits is $\lfloor 5$; $\therefore$ the expectation is

$$33333f = £34\ 14s\ 5\tfrac{1}{4}d.$$

281. (i) Retaining only the highest powers of $z$, (for the others will vanish in comparison,) we have

$$(4x+1)\,4 = 5\,(3x+1), \text{ or } x = 1.$$

(ii) Assume the fraction $= m$ for all values of $z$; then

$$(2-3m)z^2 + \{(x-a) - m(y-b)\}z + \{2b(x-2c) - 3ma(y-3c)\} = 0,$$

where we must have $2 - 3m = 0$, $(x-a) - m(y-b) = 0$, $2b(x-2c) - 3ma(y-3c) = 0$, whence $m, x, y$.

282. Since each must be div. by $x + c$, we have

$$x^2 + ax + b = (x + c)(x + a - c), \quad x^2 + a'x + b' = (x + c)(x + a' - c);$$

$$\text{hence } b = (a - c)c, \quad b' = (a' - c)c,$$

$$\text{and } \therefore \text{ L. C. M.} = (x^2 + ax + b)(x + a' - c) = \&\text{c.},$$

putting for $b$ its value.

283. (ii) Expanding and reducing, we get

$$4(a^2 - b^2)^2 c^2d^2 + 4a^2b^2(c^2 - d^2)^2 + (a^2 - b^2)^2(c^2 - d^2)^2 + 16a^2b^2c^2d^2$$
$$= \{(a^2 - b^2)^2 + 4a^2b^2\}\{(c^2 - d^2)^2 + 4c^2d^2\} = (a^2 + b^2)^2(c^2 + d^2)^2.$$

284. Putting it $= m$, we get $mx^4 - \{(m + 1)a^2 - b^2\}x^2 + a^4 = 0$, where the limiting values of $m$ are those which give

$$\{(m + 1)a^2 - b^2\}^2 = 4ma^4, \text{ or } (m + 1)a^2 - b^2 = \pm 2a^2\sqrt{m},$$

$$\text{whence } (\sqrt{m} \mp 1)^2 = \frac{b^2}{a^2}, \text{ and } m = \left(\frac{a \pm b}{a}\right)^2.$$

285. $A$'s 1st chance $= \frac{3}{8}$; $A$'s 2nd chance, (contingent on each of them drawing black the first time,) $= (\frac{5}{8} \times \frac{4}{7} \times \frac{3}{6}) \times \frac{3}{5}$; $A$'s total chance=sum of these $= \frac{27}{56}$; so $B$'s$= (\frac{5}{8}) \times \frac{3}{7} + (\frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5}) \times \frac{3}{4} = \frac{18}{56}$, and $C$'s $= 1 - \frac{27}{56} - \frac{18}{56} = \frac{11}{56}$.

286. (1) $x^2 + x + y = \square$, (2) $y^2 + y + x = \square$: put $x^2 + x + y = (x + m)^2$;

then $x = \dfrac{m^2 - y}{1 - 2m}$; $\therefore$ from (2)

$$(y^2 + y)(1 - 2m)^2 + (m^2 - y)(1 - 2m) = \square = \{(1 - 2m)y + n\}^2 \text{ suppose};$$

$$\therefore y = \frac{m^2(1 - 2m) - n^2}{2(1 - 2m)(m + n)}:$$

$$\text{put } m = 1; \text{ then } y = \frac{n^2 + 1}{2(n + 1)}, \quad x = \frac{n^2 - 2n - 1}{2(n + 1)};$$

if $n = 3$, $x = \frac{1}{4}$, $y = \frac{5}{4}$.

287. The scale of relation is $1 - x - 2x^2 = (1 + x)(1 - 2x)$; proceed as in [181].

288. In (i), $\dfrac{(r + 1)^{\text{th}}}{r^{\text{th}}} = \dfrac{x + r - 1}{r}x$, which is $\gtrless 1$, according as $(x - 1)x \gtrless r(1 - x)$, which is the case (for all values of $r$) according as $x \gtrless 1$.

In (ii), $\dfrac{(r + 1)^{\text{th}}}{r^{\text{th}}} = \dfrac{x - r + 1}{r}x$, which is $< 1$, if $\dfrac{x + 1}{r} < 1 + \dfrac{1}{x}$, or $r > x$.

289. If possible, let $p^2N^2 = an + r$, $q^2N^2 = bn + r$; then $(p^2 - q^2) N^2 = (a - b) n$, and is ∴ div. by $n$, which cannot be, since $p^2 - q^2 = (p + q)(p - q)$, where $p$ and $q$ are each $< \frac{1}{2}(n - 1)$ by supposition, and $N$, and ∴ $N^2$, is also prime to $n$.

290. The number of ways, in which it might fall *head* three times successively in five trials, is denoted by *aaaaa, aaaab, aaaba, abaaa, baaaa, aaabb, bbaaa, baaab,* viz. 8 times; ∴ chance of this $= \frac{8}{32} = \frac{1}{4} =$ ditto for its falling *tail;* ∴ chance of *not* falling either head or tail three times successively is $\frac{1}{2}$, and chance required $= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{32}$.

291. (i) Let $a, b, c, d$, be all real and arranged in order; then $x$ is plainly possible:

(ii) Let *two* be imaginary, $a$ and $b$, $= m \pm n\sqrt{-1}$;

then $(a-c)(b-c) = (m-c+n\sqrt{-1})(m-c-n\sqrt{-1}) = (m-c)^2 + n^2$;

$(a-d)(b-d) = (m-d)^2 + n^2$, and also $ab = m^2 + n^2$; ∴ &c.

(iii) Let $a$ and $b = m \pm n\sqrt{-1}$, $c$ and $d = m' \pm n'\sqrt{-1}$;

$$\therefore a-c = (m-m') + (n-n')\sqrt{-1},\ b-d = (m-m') - (n-n')\sqrt{-1},$$
$$a-d = (m-m') + (n+n')\sqrt{-1},\ b-c = (m-m') - (n+n')\sqrt{-1};$$

∴ product $= \{(m-m')^2 + (n-n')^2\}\{(m-m')^2 + (n+n')^2\}$,

$$ab = m^2 + n^2,\ cd = m'^2 + n'^2;\ \therefore \text{\&c.}$$

292. (i) $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = 0$, whence $x = \frac{1}{2}$, $y = \frac{1}{2}$:

(ii) $b\sqrt{x} + b\sqrt{y} = a\sqrt{2}\sqrt{x} - a\sqrt{3}\sqrt{y}$, whence $\dfrac{\sqrt{x}}{\sqrt{y}} = \dfrac{a\sqrt{3}+b}{a\sqrt{2}-b}$;

$$\therefore (85)\ \frac{\sqrt{x}-\sqrt{y}}{\sqrt{2}\sqrt{x}+\sqrt{3}\sqrt{y}} = \frac{(a\sqrt{3}+b)-(a\sqrt{2}-b)}{\sqrt{2}(a\sqrt{3}+b)+\sqrt{3}(a\sqrt{2}-b)} = \text{\&c}$$

293. Let the series be $\dfrac{1}{p}, \dfrac{1}{p+q}, \dfrac{1}{p+2q}$, &c; then

$$\frac{1}{p+(n-1)q} - \frac{1}{p+nq} \text{ or } \frac{q}{(p-q+nq)(p+nq)} = \frac{1}{an^2+bn+c}:$$

get rid of fractions, and equate coefficients of $n^2$, &c.

295. They might fall an *even* number of heads, either *two* together, *four* together, &c. or $n$ together, that is in $C_2 + C_4 +$ &c $+ C_n$ different ways, (where $C_0$, $C_1$, &c. are the coeff^s^ of $(1 + x)^n$,) or in $2^{n-1} - 1$ ways: but (since they might fall *no* head, *one* head, *two* heads, &c. or $n$ heads) the n° of ways in which they might *possibly* fall will be $C_0 + C_1 +$ &c $+ C_n = 2^n$; hence the chance of their falling an even n° of heads $= (2^{n-1} - 1) \div 2^n$, and the odds against it are $2^n - (2^{n-1} - 1) : 2^{n-1} - 1 =$ &c.

296. Let $P$ = original property: then $P_1$ (at end of first year)

$$= \{1 + (1 - m)\,r\}\,P,\ \ P_2 = \{1 + (1 - 2m)\,r\}\,P_1,\ \&c.,$$
$$P_{2p} = \{1 + (1 - 2pm)\,r\}\,P_{2p-1} = 0;$$
$$\therefore 1 + r = 2pmr, \text{ and } \therefore P_p = \{1 + (1 - pm)\,r\}\,P_{p-1} = pmr\,P_{p-1}.$$

297. Let $N + a = N_1^2$, $N - b = (N_1 - 1)^2$; $\therefore a + b = 2N_1 - 1$, or $N_1 = \frac{1}{2}(a + b + 1)$; $\therefore N + a = \frac{1}{4}\{(a + b)^2 + 2(a + b) + 1\}$, and thence $N - ab = \frac{1}{4}\{(a - b)^2 - 2(a - b) + 1\}$.

298. Given $x - y + 1 = \square$, $x + 1 = \square$, $x + y + 1 = \square$, $y + 1 = \square$: then, since the first three are in A. P., put [Ex. **43**, PROB. 2].

$$x - y + 1 = (p^2 - q^2 - 2pq)^2,\ \ x + 1 = (p^2 + q^2)^2,\ \ (x + y + 1) = (p^2 - q^2 + 2pq)^2;$$

then $y + 1 = 1 + 4p^3q - 4pq^3 = (1 + 2p^3q)^2$, suppose;

$\therefore q = -p^5$, and $y = (1 - 2p^9)^2 - 1 = 4p^8(p^8 - 1)$, and $x + 1 = p^4(p^8 + 1)^2$, which, as well as the values of $y + 1$, &c., will be found to be still squares, if we put $m$ for $p^2$, so that

$$x = m^2(m^4 + 1)^2 - 1,\ \ y = 4m^4(m^4 - 1).$$

If $m = 2$, then $x = 1155$, $y = 960$, and the squares are

$$34^2,\ 31^2,\ 46^2,\ 14^2.$$

299. $(1 - x)^{-2} = \{1 - (1 + n^{-1})\}^{-2} = (-n)^2 = n^2$.

300. Chance of throwing sixes in one cast $= \frac{1}{36} = p$; $\therefore$ chance of throwing sixes at least once in $x$ throws

$$= p^x + xp^{x-1}q + \&c. + xpq^{x-1} = 1 - q^x = P, \text{ suppose:}$$
$$\therefore \text{if } P = \tfrac{1}{2},\ q^x = \tfrac{1}{2}, \text{ and } x = \frac{\log \frac{1}{2}}{\log \frac{35}{36}} = \frac{\log 2}{\log 36 - \log 35} = 24.6;$$
$$\therefore \text{if } x = 25,\ P > \tfrac{1}{2}.$$

301. $r^{10} + 2r^9 + 3r^8 + \&c.$ is div. by $r^4 + 2r^3 + 3r^2 + 2r + 1$, whatever $r$ may be, since it $= (r^4 + 2r^3 + \&c.) \times (r^6 + 2r^3 + 1)$.

302. In $(\sqrt{2})^{12\sqrt{6}}\,(1 + \sqrt{\frac{3}{2}})^{12\sqrt{6}}$, we have by [95]

$$r > (12\sqrt{6} + 1)\,\frac{\sqrt{3}}{\sqrt{3} + \sqrt{2}} > 35\sqrt{6} - 69 > 16.715.$$

303. Let $x^2 + y^2 - 1 = \square$, $x^2 - y^2 - 1 = \square = n^2$, suppose; then

$$y^2 + 1 + n^2 = \square = (y - 1 + n)^2, \text{ suppose;}$$
$$\therefore y = \frac{n}{1 - n} = \frac{1 - m}{m}, \text{ writing } m \text{ for } 1 - n;$$
$$\therefore x^2 + \frac{1 - 2m}{m^2} = \square, \text{ or } m^2x^2 + 1 - 2m = \square = (mx - p)^2;$$
$$\therefore x = \frac{p^2 + 2m - 1}{2mp}; \text{ if } m = \tfrac{1}{10},\ y = 9, \text{ and } x = \frac{5p^2 - 4}{p} = 8, \text{ if } p = 2.$$

304. The scale of relation being $1 - 4x + 6x^2 - 4x^3 + x^4 = (1-x)^4$, find $S$ by [182].

305. The number of ways in which four cards can be drawn is $\frac{52.51.50.49}{1.2.3.4}$; the number of ways in which the required drawing can be made is 4.3.2.1; &c.

306. $$m\phi(n, m) = 1 - n\frac{m}{m+p} + \frac{n-1}{2}\cdot\frac{m}{m+2p} - \&c.$$
$$= 1 - n\left(1 - \frac{p}{m+p}\right) + n\frac{n-1}{2}\left(1 - \frac{2p}{m+2p}\right) - \&c.$$
$$= (1-1)^n + np\,\phi(n-1, m+p);$$
$$\therefore \phi(n, m) = \frac{np}{m}\phi(n-1, m+p) = \frac{np}{m}\cdot\frac{(n-1)p}{m+p}\phi(n-2, m+2p)$$
$$= \&c. = \frac{n\,\lfloor p^n}{m(m+p)\ldots(m+np)}.$$

307. $N = a^2 + b^2 + c^2 + \&c. + k^2$: now $2(a^2+b^2) = (a+b)^2 + (a-b)^2$ = sum of two squares, and
$$2(n-1)N = 2(n-1)a^2 + 2(n-1)b^2 + 2(n-1)c^2 + \&c.$$
$$= 2(a^2+b^2) + 2(a^2+c^2) + 2(b^2+c^2) + \&c.,$$
where there may be formed $\frac{1}{2}n(n-1)$ pairs, each of which may be resolved into the sum of two squares.

308. $$\frac{1}{1-xz} + \frac{1}{1-x} + \frac{1}{1-xz^{-1}} = \frac{3 + (x^2-2x)(z+1+z^{-1})}{1 + (x^2-x)(z+1+z^{-1}) - x^3} = \frac{3}{1-x^3},$$
since $z + 1 + z^{-1} = 0$;
$$\therefore (1 + xz + x^2z^2 + \&c.) + (1 + x + x^2 + \&c.) + (1 + xz^{-1} + x^2z^{-2} + \&c.)$$
$$= 3(1 + x^3 + x^6 + \&c.),$$
whence coeff. of $x^r = z^r + 1 + z^{-r} = 0$, unless $r$ be a multiple of 3, when this sum will $= 3$. Now coefficient of $x^r$ in given quantity is $\frac{1}{3}\{C_r(z^r + 1 + z^{-r})\}$, which $\therefore = 0$, unless $r$ be a multiple of 3, when it $= C_r =$ coefficient of $x^r$ in $(1+x)^n$.

309. Put $S' = 1 + \frac{x}{3} + \frac{x^2}{5} + \&c. + \frac{x^n}{2n+1}$, and mult. both sides by $1 - 3x$; then $S'(1-3x)$
$$= 1 - x(3 - \tfrac{1}{3}) - x^2(\tfrac{3}{3} - \tfrac{1}{5}) - x^3(\tfrac{3}{5} - \tfrac{1}{7}) - \&c. - x^n\left(\frac{3}{2n-1} - \frac{1}{2n+1}\right) - \frac{3x^{n+1}}{2n+1}$$
$$= 1 - \frac{8}{1.3}x - \frac{12}{3.5}x^2 - \frac{16}{5.7}x^3 - \&c. - \frac{4n+4}{(2n-1)(2n+1)}x^n - \frac{3x^{n+1}}{2n+1};$$

put $1 - 3x = 0$, or $x = \frac{1}{3}$; then

$$0 = 1 - 4S - \frac{1}{(2n+1)\,3^n}, \text{ or } S = \frac{1}{4} - \frac{1}{4\,(2n+1)\,3^n}.$$

310. It is true if the first reporter spoke truly, and all the others also, or if he spoke falsely, and all the others also; the probability is $\therefore$ by [219 Ex. 2]

$$\frac{v}{v+m} \cdot \frac{v^p}{v^p + m^p} + \frac{m}{v+m} \cdot \frac{m^p}{v^p + m^p}.$$

311. (i) Put $z = \log_e (1+n)$; $\therefore 1 + n = e^z = 1 + z + \frac{1}{2}z^2 + \text{\&c}$:

(ii) $\{(x+y)^{\frac{2}{3}} - (x-y)^{\frac{2}{3}}\}^3 = (x+y)^2 - (x-y)^2 - 3\,(x^2-y^2)^{\frac{2}{3}}\,\{(x+y)^{\frac{2}{3}} - (x-y)^{\frac{2}{3}}\}$;

$\therefore 4xy - 3\,(x^2-y^2)^{\frac{2}{3}} = \{(x+y)^{\frac{2}{3}} - (x-y)^{\frac{2}{3}}\}^3 + 3\,(x^2-y^2)^{\frac{2}{3}}\,\{(x+y)^{\frac{2}{3}} - (x-y)^{\frac{2}{3}} - 1\}$

$= z^3 + 3\,(x^2-y^2)^{\frac{2}{3}}\,(z-1)$, suppose, which is $\gtrless 1$, as $z \gtrless 1$.

313. Let the n° of areas be $2m+1$: then $r_1^2 = \dfrac{a^2}{2m+1}$, $r_2^2 = \dfrac{2a^2}{2m+1}$, &c;

$$\therefore a = p\,(r_{m+1} - r_m) = \frac{pa}{\sqrt{(2m+1)}}\,\{\sqrt{(m+1)} - \sqrt{m}\};$$

$$\therefore \frac{2m+1}{p^2} = \{\sqrt{(m+1)} - \sqrt{m}\}^2 = 2m + 1 - 2\sqrt{(m^2+m)},$$

$$\text{or } (2m+1)\,\frac{p^2-1}{p^2} = 2\sqrt{(m^2+m)} = \sqrt{\{(2m+1)^2 - 1\}};$$

squaring, we get $2m + 1 = \dfrac{p}{\sqrt{(2p^2-1)}}$.

314. (2) + (3) gives

$$b^3 + c^3 = x\,\{(y+z)^2 - 2yz\} + yz\,(y+z) = x\left(\frac{a^6}{x^4} - \frac{2abc}{x}\right) + \frac{abc}{x} \cdot \frac{a^3}{x^2};$$

$$\therefore x^3 = \frac{a^4\,(a^2+bc)}{a^3 + 2abc + b^3}:$$

(2) – (3) gives $\dfrac{b^3 - c^3}{y - z} = x\,(y+z) + yz = \dfrac{a^3}{x} + \dfrac{abc}{x}$,

$$\text{or } y - z = \frac{b^3 - c^3}{a\,(a^2+bc)}\,x, \text{ and } y + z = \frac{a^3}{x^2};$$

hence $4abc = 4xyz = x\,\{(y+z)^2 - (y-z)^2\} = \dfrac{a^6}{x^3} - \dfrac{(b^3-c^3)^2}{a^2\,(a^2+bc)^2}\,x^3$

$= (\text{by putting for } x^3 \text{ its value})\ \dfrac{a^2\,\{(b^3 + 2abc + c^3)^2 - (b^3 - c^3)^2\}}{(a^2+bc)\,(b^3 + 2abc + c^3)}$

$$= \frac{a^2\,(2b^3 + 2abc)\,(2c^3 + 2abc)}{(a^2+bc)\,(b^3 + 2abc + c^3)};$$

$\therefore (a^2+bc)\,(b^3 + 2abc + c^3) = a\,(b^2+ac)\,(c^2+ab)$, whence $a^3 + b^3 + c^3 + abc = 0$.

315. Using $b$, $c$, $d$, for the chance of *A's winning*, and $b'$, $c'$, $d'$, for that of his *losing*, with $B$, $C$, $D$, respectively, we have the chance required in (i) $= bcd' + bc'd + b'cd = \frac{1}{4} \times \frac{3}{5} \times \frac{3}{7} + \&c$, in (ii) $= bcd + bcd' + bc'd + b'cd$.

316. $f(p, q) =$ coeff. of $x^q$ in $\left(\frac{1-x^{n+1}}{1-x}\right)^p$; hence the given series

$$= \text{coeff. of } x^q \text{ in } \frac{1-x^{n+1}}{1-x} - \tfrac{1}{2} \text{ coeff. of } x^{q-1} \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^2 + \&c.$$

$$= \text{coeff. of } x^q \text{ in } \frac{1-x^{n+1}}{1-x} - \tfrac{1}{2}x\left(\frac{1-x^{n+1}}{1-x}\right)^2 + \tfrac{1}{3}x^2\left(\frac{1-x^{n+1}}{1-x}\right)^3 - \&c.$$

$$= \text{coeff. of } x^q \text{ in } \frac{1}{x}\log_e\{1 + x(\frac{1-x^{n+1}}{1-x})\} \text{ or in } \frac{1}{x}\log_e \frac{1-x^{n+2}}{1-x}$$

$$\text{that is, in } \frac{1}{x}\{\log_e(1-x^{n+2}) - \log_e(1-x)\},$$

$$\text{or in } \frac{1}{x}\{x + \frac{x^2}{2} + \&c + \frac{x^{q+1}}{q+1} + \&c\} - \frac{1}{x}\{x^{n+2} + \frac{x^{2(n+2)}}{2} + \&c\},$$

and is $\therefore \frac{1}{q+1}$, unless $q+1$ is some multiple of $n+2$, such as $m(n+2)$, when it $= \frac{1}{q+1} - \frac{1}{m}$.

317. For $a_1$, $a_2$, &c. use $a$, $ar$, &c., and for $x_1$, $x_2$, &c. use $x$, $x+b$, &c.; then

$$a^x . (ar)^{x+b} \ldots (ar^{n-1})^{x+(n-1)b} = a^{x+(x+b)+\&c} \times r^{(x+b)+2(x+2b)+\&c},$$

where the index of $a$ is $(x + x_n)\frac{n}{2}$, and that of $r$ is

$$\frac{n(n-1)}{2}x + \frac{n(n-1)(2n-1)}{1.2.3}b = \frac{n(n-1)}{2}x + \frac{n(2n-1)}{6}(x_n - x):$$

put now for $r$ its value $\left(\frac{a_n}{a}\right)^{\frac{1}{n-1}}$.

318. Put $S' = \frac{1}{1.2} + \frac{x}{2.3} + \frac{x^2}{3.4} + \&c + \frac{x^n}{(n+1)(n+2)}$,

and mult. by $1 - 2x$;

$$\therefore S'(1-2x) = \frac{1}{1.2} - x(\frac{2}{1.2} - \frac{1}{2.3}) - x^2(\frac{2}{2.3} - \frac{1}{3.4}) - \&c.$$

$$- x^n\left(\frac{2}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right) - \frac{2x^{n+1}}{(n+1)(n+2)}.$$

$$= \frac{1}{1.2} - \frac{5}{1.2.3}x - \frac{6}{2.3.4}x^2 - \&c - \frac{n+4}{n(n+1)(n+2)}x^n - \frac{2x^{n+1}}{(n+1)(n+2)};$$

(i) put $x = \frac{1}{2}$; $\therefore\ S = \frac{1}{2} - \frac{1}{(n+1)(n+2)2^n}$:

(ii) put $x = 1$; $\therefore\ -S' = \frac{1}{2} - S - \frac{2}{(n+1)(n+2)}$,

where we must first find $S'$:

now $S' = \frac{1}{1.2} + \frac{1}{2.3} + \&c = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \&c + (\frac{1}{n+1} - \frac{1}{n+2})$

$$= 1 - \frac{1}{n+2} = \frac{n+1}{n+2}; \quad \therefore\ S = \frac{3}{2} - \frac{n+3}{(n+1)(n+2)}.$$

319. Since $x^2 + y^2$ becomes $(p^2 + q^2)^2$ by putting $x = p^2 - q^2$, $y = 2pq$, so that $x \pm y\sqrt{-1} = (p \pm q\sqrt{-1})^2$, this suggests that $ax^2 + by^2$ may be converted to $(ap^2 + bq^2)^n$ by putting for $x$ and $y$ their values deduced from $x\sqrt{a} \pm y\sqrt{b}\sqrt{-1} = (p\sqrt{a} \pm q\sqrt{b}\sqrt{-1})^n$, which supposition, however, we can only make when $n$ is an odd number, since otherwise we should have the terms on the right-hand side partly rational, and the others involving $\sqrt{(ab)}\sqrt{-1}$. Hence, by expanding and adding, &c, we get the values of $x$ and $y$.

320. Put one of the *two* persons on a given chair; on each side of him seat one of the *thirteen* others, who may be taken, two and two, in 78 ways, and, besides, the two of any one of these pairs may be seated on each side of him in 2 ways, so that there are altogether $2 \times 78$ or 156 ways in which this may be done: now seat the remaining *twelve* in any order whatever, which may be done in $\lfloor 12$ ways; also the chair, assumed to be given, may be any one of the 15. Hence the n° of ways in which the proposed arrangement can be made, so that the two shall not sit next each other, is $15 \times 156 \lfloor 12$; but the total n° of permutations of them is $\lfloor 15$: hence the chance $= \frac{6}{7}$. Generally, in the same way, the chance

$$= \frac{n(n-2)(n-3)\lfloor n-3}{\lfloor n} = \frac{n-3}{n-1}.$$

321. We know [158] that this is true up to $\frac{p_n}{q_n}$: now let

$$x = \frac{mp_n + p_{n-1}}{mq_n + q_{n-1}}, \quad \frac{p}{q} = \frac{m'p_n + p_{n-1}}{m'q_n + q_{n-1}};$$

$$\text{then } x \sim \frac{p}{q} = \frac{m \sim m'}{(mq_n + q_{n-1})(m'q_n + q_{n-1})},$$

which becomes less and less as $m'$ increases, so long as $m' < m$, that is up to $\frac{p_{n+1}}{q_{n+1}}$, for which $m'$ = the integer in $m$. It is plain also that, for all such values of $m'$, $\frac{p}{q}$ lies on the same side of $x$ as $\frac{p_{n+1}}{q_{n+1}}$, and *beyond* the latter; so that $x$, which lies between $\frac{p_{n+1}}{q_{n+1}}$ and $\frac{p_n}{q_n}$ must, *a fortiori*, lie between $\frac{p}{q}$ and $\frac{p_n}{q_n}$. Now suppose $\frac{r}{s}$ to lie between $\frac{p}{q}$ and $x$, and therefore, *a fortiori*, between $\frac{p}{q}$ and $\frac{p_n}{q_n}$;

$$\therefore \frac{r}{s} \sim \frac{p_n}{q_n} < \frac{p_n}{q_n} \sim \frac{p}{q} < \frac{p_n}{q_n} \sim \frac{m'p_n + p_{n-1}}{m'q_n + q_{n-1}} < \frac{\pm 1}{q_n q}, \text{ or } rq_n \sim p_n s < \frac{s}{q},$$

which is impossible, if $s < q$, since $p_n$, $q_n$, $r$, $s$, are integers.

322. $\frac{x^m}{(x-a)(x-b)\dots(x-h)} = \frac{A_1}{x-a} + \frac{A_2}{x-b} + \text{\&c.}$, where we have, by [172 Ex. 5] $A_1 = \frac{a^m}{(a-b)(a-c)\dots(a-h)}$, $A_2 = \text{\&c.}$; whence, by reducing to a common denominator and equating coefficients, we have $0 = \text{coeff. of } x = A_1 + A_2 + \text{\&c.} = \text{\&c.}$

323. $\log_e x = m \log_e x^{\frac{1}{m}} = m \log_e \{1 + (x^{\frac{1}{m}} - 1)\} = m\{(x^{\frac{1}{m}} - 1) - \frac{1}{2}(x^{\frac{1}{m}} - 1)^2 + \text{\&c}\}$,

$= m(x^{\frac{1}{m}} - 1)$ nearly, if $m$ be a large number, so that $x^{\frac{1}{m}}$ may differ little from 1: hence $\log_e x = 2^n (x^{\frac{1}{2^n}} - 1)$, if $n$ be great.

Now $(x-1) = (x^{\frac{1}{2}} + 1)(x^{\frac{1}{2}} - 1) = (x^{\frac{1}{2}} + 1)(x^{\frac{1}{4}} + 1)(x^{\frac{1}{4}} - 1)$

$= (x^{\frac{1}{2}} + 1)(x^{\frac{1}{4}} + 1)\dots(x^{\frac{1}{2^n}} + 1)(x^{\frac{1}{2^n}} - 1)$; $\therefore \log_e x = 2^n (x^{\frac{1}{2^n}} - 1)$

(if $n$ be very great) $= (x-1)\frac{2}{x^{\frac{1}{2}} + 1} \cdot \frac{2}{x^{\frac{1}{4}} + 1} \dots$.

324. This question reduces to $x + y = m^2$, $x - y = n^2$, and $x^2 + xy + y^2 = \square$;

$$\therefore x = \tfrac{1}{2}(m^2 + n^2), \quad y = \tfrac{1}{2}(m^2 - n^2),$$

$$x^2 + xy + y^2 = \tfrac{1}{4}(3m^4 + n^4) = \tfrac{1}{4}m^4(3 + z^4), \text{ suppose, } = \square:$$

$$\text{put } z^4 + 3 = (z^2 - m)^2; \quad \therefore z^2 = \frac{m^2 - 3}{2m} = \frac{2m(m^2 - 3)}{4m^2} = \square;$$

hence $2m^3 - 6m^2 = \square$, which has one solution, $m = -1$; put $m = -1 + r$; then $2r^3 - 6r^2 + 4 = \square = 4$, suppose (assuming by the rule in [207], where $2mp = c = 0$, and $\therefore n = 0$); $\therefore r = 3$, $m = 2$, $z = \frac{1}{2}$, $m = 2n$; and in order to have $x$ and $y$ integers, put $n = 2$; then $m = 4$, $x = 10$, $y = 6$.

325. If side of square face $= 1$, it is easily seen that side of octagon $= \sqrt{2} - 1$, and sum of areas of eight triangular faces $= 2\sqrt{3}(\sqrt{2} - 1)^2$, sum of areas of six octagonal faces $= 12(\sqrt{2} - 1)$; $\therefore$ chances are as

$$\sqrt{3}(\sqrt{2} - 1) : 6 :: \sqrt{2} - 1 : 2\sqrt{3}.$$

326. $(a + b) = (s + p)^{\frac{1}{2}p}$; $\therefore P = (s + p)^{\frac{p}{2}}$, and

$$P . P^{\frac{1}{2}} . P^{\frac{1}{4}} . \&c. = (s + p)^{\frac{p}{2}(1 + \frac{1}{2} + \&c.)} = (s + p)^p = \&c.$$

327. Let $S = x + 4x^2 + 9x^3 + 16x^4 + \&c$;

$\therefore Sx = x^2 + 4x^3 + 9x^4 + \&c$,

and $S(1 - x) = x + 3x^2 + 5x^3 + 7x^4 + \&c$;

$\therefore Sx(1 - x) = x^2 + 3x^3 + 5x^4 + \&c$,

and $S(1 - x)^2 = x + 2x^2 + 2x^3 + 2x^4 + \&c = x + \dfrac{2x^2}{1 - x} = \dfrac{x + x^2}{1 - x}$;

$\therefore S = \dfrac{x + x^2}{(1 - x)^3}$: put $x = \frac{1}{2}$, or $\frac{1}{3}$, or $-\frac{1}{5}$, attending to the *sign* in the last case.

328. The n° of triangular faces $= \dfrac{6.5.4}{1.2.3} = 20$, on each of which may be erected six different pyramids with the other three lines; $\therefore$ 120 = whole n° thus formed: but each pyramid is produced thus four times, (as appears by considering that any one pyramid may be placed on a table in four different ways;) $\therefore$ n° required = 30.

329. $\dfrac{pm}{a^2} = -\dfrac{qn}{b^2}$, $\dfrac{m^2}{a^2} = \dfrac{n^2}{b^2}$; $\therefore \dfrac{p}{m} = -\dfrac{q}{n}$, and $\dfrac{x}{m} + \dfrac{y}{n} = \dfrac{2p}{m} = -\dfrac{2q}{n}$;

$$\text{but } \frac{m}{a} = \frac{n}{b}; \ \therefore \frac{x}{a} + \frac{y}{b} = \frac{2p}{a} = -\frac{2q}{b},$$

$$\text{and } 1 = \frac{p^2}{a^2} + \frac{q^2}{b^2} = \frac{2p^2}{a^2} = \frac{1}{2}\left(\frac{x}{a} + \frac{y}{b}\right)^2; \ \therefore \frac{x}{a} + \frac{y}{b} = \sqrt{2}.$$

330. There is an equal chance of the n° of terms being any one of the $q - p + 1$ numbers from $p$ to $q$ inclusive: hence the chance required $= \dfrac{1}{q - p + 1}\left\{\dfrac{x^p - 1}{x - 1} + \dfrac{x^{p+1} - 1}{x - 1} + \&c + \dfrac{x^q - 1}{x - 1}\right\} = \&c.$

331. Put $x = 1 + h$; then

$$(1+h)^{1+h} = (1+h)(1+h)^h = (1+h)\{1 + h^2 + \frac{h(h-1)}{1.2}h^2 + \&c\}$$

$$= (1+h)(1 + h^2 - \tfrac{1}{2}h^3 + \&c) = 1 + h + h^2 + \tfrac{1}{2}h^3 + \&c;$$

$\therefore$ for 1st App. $x^x = 1 + h = x$; for 2nd App., $= 1 + h + h^2$
$= x + (x-1)^2 = 1 - x + x^2$; &c.

333. (i) Sum after $n$ terms

$$= \frac{1}{n+1} + \frac{1}{n+2} + \&c = \frac{1}{n+1} + \&c + \frac{1}{2n} + \frac{1}{2n+1} + \&c + \frac{1}{4n} + \&c,$$

which is $> (\frac{1}{2n} + \frac{1}{2n} + \&c$ to $n$ terms$) + (\frac{1}{4n} + \frac{1}{4n} + \&c$ to $2n$ terms$)$

$$+ (\frac{1}{8n} + \frac{1}{8n} + \&c \text{ to } 4n \text{ terms}) + \&c > \frac{n}{2n} + \frac{2n}{4n} + \frac{4n}{8n} + \&c$$

$> \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \&c$, which is infinitely great:

(ii) Sum after $n$ terms

$$= \frac{1}{\lfloor\underline{n}} + \frac{1}{\lfloor\underline{n+1}} + \&c < \frac{1}{\lfloor\underline{n}}\{1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \&c\}$$

$$< \frac{1}{\lfloor\underline{n}}\left\{\frac{1}{1 - \frac{1}{n+1}}\right\} < \frac{1}{\lfloor\underline{n}} \cdot \frac{n+1}{n},$$

which diminishes indefinitely as $n$ increases.

334. $$\frac{x}{(1-x)^2 - cx} = \frac{x}{(1-x)^2}\{1 + \frac{cx}{(1-x)^2} + \frac{c^2x^2}{(1-x)^4} + \&c\}$$

$$= x\{\frac{1}{(1-x)^2} + \frac{cx}{(1-x)^4} + \frac{c^2x^2}{(1-x)^6} + \&c\},$$

where coeff. of $x^r = r\{1 + \frac{r^2-1}{2.3}c + \frac{(r^2-1)(r^2-4)}{2.3.4.5} + \&c\}$,

as will be found by expanding each term of the series by the Binomial Theorem.

335. The n° of tickets $= 1 + 2 + \&c + (n^2+1) = \frac{1}{2}(n^2+1)(n^2+2)$;
value of prizes in shillings

$$= 1 + 2^2.2 + 3^2.3 + \&c + n^2.n = 1^3 + 2^3 + \&c = \tfrac{1}{4}n^2(n+1)^2;$$

$$\therefore \text{expectation} = \frac{n^2(n+1)^2}{2(n^2+1)(n^2+2)}.$$

336. Let $x^2 + axy + by^2 = (x + my)(x + ny)$,

$x'^2 + ax'y' + by'^2 = (x' + my')(x' + ny')$,

where $m$, $n$, are roots of $z^2 - az + b = 0$: then

$(x+my)(x'+my') = xx' + m(xy'+yx') + m^2yy' = xx' + m(xy'+yx') + (am-b)yy'$, since $m^2 - am + b = 0$,

$= xx' - byy' + m(xy' + yx' + ayy') = X + mY$, suppose: similarly, $(x+ny)(x'+ny') = X + nY$; and the whole product

$$= X^2 + (m+n)XY + mnY^2 = X^2 + aXY + bY^2;$$

and so on, of course, with any number of such factors.

337. By [336], $(x^2 + 3xy + 5y^2)^2 = X^2 + 3XY + 5Y^2$, where $X = xx' - byy' = x^2 - 5y^2$, $Y = xy' + yx' + ayy' = 2xy + 3y^2$, and $x$ and $y$ may have any value; putting $y = 1$, and $x = 3, 4, 5$, &c, we get

$$X = 4,\ 11,\ 20,\ 31,\ \&c,\quad Y = 9,\ 11,\ 13,\ 15,\ \&c.$$

338. $s = a+b$, $p = ab$, $q = \frac{a}{b}$; $\therefore 1+q = \frac{a+b}{b} = \frac{s}{b}$, and $b^2 = \frac{p}{q} = \left(\frac{s}{1+q}\right)^2$;

$\therefore p^2 = s^4q^2(1+q)^{-4} = \&c$: also $(x-a)(x-b) = x^2 - sx + p$;

$$\therefore \left(1 - \frac{a}{x}\right)\left(1 - \frac{b}{x}\right) = 1 - \frac{1}{x}\left(s - \frac{p}{x}\right),$$

$$\text{and } \log\left(1 - \frac{a}{x}\right) + \log\left(1 - \frac{b}{x}\right) = \log\left\{1 - \frac{1}{x}\left(s - \frac{p}{x}\right)\right\};$$

$$\therefore -\left(\frac{a}{x} + \frac{a^2}{2x^2} + \&c + \frac{a^n}{nx^n} + \&c\right) - \left(\frac{b}{x} + \frac{b^2}{2x^2} + \&c + \frac{b^n}{nx^n} + \&c\right)$$

$$= -\left\{\frac{1}{x}\left(s - \frac{p}{x}\right) + \frac{1}{2x^2}\left(s - \frac{p}{x}\right)^2 + \&c + \frac{1}{nx^n}\left(s - \frac{p}{x}\right)^n + \&c\right\};$$

hence, equating coefficients of $\frac{1}{x^n}$, we get

$$\frac{1}{n}(a^n + b^n) = \frac{1}{n}s^n - \frac{1}{n-1}\{(n-1)s^{n-2}p\} + \frac{1}{n-2}\left\{\frac{(n-2)(n-3)}{1.2}s^{n-4}p^2\right\} - \&c.$$

339. $S = \frac{1}{3.4}\left\{\frac{1}{1.2} + \frac{1}{2.3} + \&c\right\} =$, as in Ex. 318, $\frac{1}{12}\left\{\frac{n}{n+1}\right\}$.

340. The n° of ways in which 5 balls can be taken out of the $50 = \frac{50.49.48.47.46}{1.2.3.4.5}$, and the number of ways in which the given *two* can be drawn together with 3 of the remaining $48 = \frac{48.47.46}{1.2.3}$; whence $A$'s chance of drawing them at his first drawing $= \frac{4.5}{50.49} = \frac{2}{245}$. Now, if $x = A$'s chance of success, then, since $\frac{243}{245} = A$'s chance of failing the *first* time, $\frac{243}{245}x = B$'s chance of success; $\therefore x + \frac{243}{245}x = 1$, and $x = \frac{245}{488}$; $\therefore$ odds in favor of $A$ are 245 : 243.

341. $\dfrac{x+1}{x-1} = \dfrac{1}{2}\left\{\dfrac{\sqrt{x}-1}{\sqrt{x}+1} + \dfrac{\sqrt{x}+1}{\sqrt{x}-1}\right\} = \dfrac{1}{2}\cdot\dfrac{\sqrt{x}-1}{\sqrt{x}+1} + \dfrac{1}{4}\left\{\dfrac{\sqrt[4]{x}-1}{\sqrt[4]{x}+1} + \dfrac{\sqrt[4]{x}+1}{\sqrt[4]{x}-1}\right\}$

$= \&\text{c} + \dfrac{1}{2^n}\cdot\dfrac{x^{\frac{1}{2^n}}-1}{x^{\frac{1}{2^n}}+1} + \dfrac{1}{2^n}\cdot\dfrac{x^{\frac{1}{2^n}}+1}{x^{\frac{1}{2^n}}-1}$, where the last term becomes (ultimately, when $n$ is very great) $\dfrac{2}{2^n\,(x^{\frac{1}{2^n}}-1)} =$, as in Ex. 323, $\dfrac{2}{\log_e x}$; $\therefore\ \dfrac{1}{\log_e x} = \dfrac{1}{2}\cdot\dfrac{x+1}{x-1} + \&\text{c}.$

342. Put $v = x^{\frac{1}{2^r}}$; then, if $u_r$ denote the $r$th term, we have

$$\frac{u_r}{u_{r-1}} = \frac{1}{2^r}\cdot\frac{v-1}{v+1} \div \frac{1}{2^{r-1}}\cdot\frac{v^2-1}{v^2+1} = \frac{1}{2}\cdot\frac{v^2+1}{(v+1)^2} > \frac{1}{2}\cdot\frac{v^2+1}{2\,(v^2+1)} > \frac{1}{4}:$$

and $\dfrac{(u_{r-1})^2}{u_{r-2}} = \dfrac{1}{4}\cdot\dfrac{(v^2-1)\,(v+1)^2}{(v^2+1)^2}$, $u_r = \dfrac{1}{4}\cdot\dfrac{v^4-1}{v^4+1}$; $\therefore\ \dfrac{u^2_{r-1}}{u_{r-2}} > u_r$, if $(v+1)^2\,(v^4+1) > (v^2+1)^3$, or $v^4 - v^3 > v - 1$, or $v > 1$.

343. Consider the expressions

$(x+c)\,(x+c^2)\ldots(x+c^n) = x^n + P_1x^{n-1} + \&\text{c} + P_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$

$(x+c)\,(x+c^2)\ldots(x+c^{n-1}) = x^{n-1} + Q_1x^{n-1} + \&\text{c} + Q_{n-1} \ldots\ldots\ldots\ldots\ldots\ldots(2),$

$(x+c^2)\,(x+c^3)\ldots(x+c^n) = x^{n-1} + cQ_1x^{n-2} + c^2Q_2x^{n-3} + \&\text{c} + c^{n-1}Q_{n-1}\ldots(3),$

where (3) is formed from (2) by changing $x$ into $\dfrac{x}{c}$, and multiplying by $c^{n-1}$.

Now $(1) = (2)\times(x+c^n)$, whence, equating coeffˢ, $P_m = Q_m + c^nQ_{m-1}$;
and $(1) = (3)\times(x+c)$, $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $P_m = c^mQ_m + c^mQ_{m-1}$;

$$\therefore\ Q_{m-1} = \frac{c^m-1}{c^m\,(c^n-1)}\,P_m, \text{ or } Q_m = \frac{c^{m+1}-1}{c^{m+1}\,(c^n-1)}\,P_{m+1};$$

$$\therefore\ P_m = c^mQ_m + c^mQ_{m-1} = \frac{c^{m+1}-1}{c\,(c^n-1)}\,P_{m+1} + \frac{c^m-1}{c^n-1}\,P_m,$$

$$\text{or } P_{m+1} = \frac{c\,(c^n-c^m)}{c^{m+1}-1}\,P_m = \frac{c^{m+1}\,(c^{n-m}-1)}{c^{m+1}-1}\,P_m:$$

hence $P_m = c^m.\dfrac{c^{n-m+1}-1}{c^m-1}\,P_{m-1} = c^m.c^{m-1}.\dfrac{(c^{n-m+1}-1)\,(c^{n-m+2}-1)}{(c^m-1)\,(c^{m-1}-1)}\,P_{m-2} = \&\text{c},$
till at last we come to the factor $P_1$, which

$$= c + c^2 + \&\text{c} + c^n = c\,.\,\frac{c^n-1}{c-1};$$

whence, (since $c^m.c^{m-1}.c^{m-2}\ldots c = c^{\frac{1}{2}m(m+1)}$) the Theorem is plainly demonstrated.

344. $ab < \left(\frac{a+b}{2}\right)^2$, $cd < \left(\frac{c+d}{2}\right)^2$; $\therefore abcd < \left(\frac{a+b}{2} \cdot \frac{c+d}{2}\right)^2$:

but $\frac{a+b}{2} \cdot \frac{c+d}{2} < \left\{\frac{\frac{1}{2}(a+b) + \frac{1}{2}(c+d)}{2}\right\}^2 < \left(\frac{a+b+c+d}{4}\right)^2$;

$\therefore abcd < \left(\frac{a+b+c+d}{4}\right)^4$; and thus, generally,

$$abcd \ldots (p \text{ factors}) < \left(\frac{a+b+c+\&c}{p}\right)^p, \text{ if } p = 2^m.$$

Now let $p = n + r$, and let $\frac{a+b+c+\&c\ (n \text{ terms})}{n} = k$, and let each of the remaining $r$ terms $= k$; we have then

$$abcd \ldots (n \text{ factors}) \times k^r < \left(\frac{nk+rk}{n+r}\right)^{n+r} < k^{n+r},$$

$$\text{or } abcd \ldots (n \text{ factors}) < k^n < \left(\frac{a+b+c+\&c}{n}\right)^n,$$

where for $a$, $b$, $c$, &c. write $a^n$, $b^n$, $c^n$, &c.

Hence $1 + 2 + 3 + \&c + n$, or $\frac{1}{2}n(n+1) > n\sqrt[n]{(1.2.3 \ldots n)}$, or $1.2.3 \ldots n < \{\frac{1}{2}(n+1)\}^n$.

345. Chance for *one* trial $= \frac{1}{2}(e+e') + \frac{1}{2}(e-e') = e$; chance for *two* trials $= \frac{1}{2}(e+e')^2 + \frac{1}{2}(e-e')^2 = e^2 + e'^2$; chance for $n$ trials $= \frac{1}{2}(e+e')^n + \frac{1}{2}(e-e')^n = e^n + \&c$, and $\therefore > e^n$.

346. From (1) and (2),

$$(y-vx)^2 + rx + s = y^2 - 2vyx + v^2(y-vx) + rx + s = 0,$$

$$\text{whence } x = \frac{y^2 + v^2y + s}{2vy + v^3 - r}:$$

then (2) gives

$$(y^2 + v^2y + s)^2 + v(y^2 + v^2y + s)(2vy + v^3 - r) = (2vy + v^2 - r)^2 y,$$

which manifestly wants the second term; also coeff. of $y = 0$, if $2v^2s + v^3(v^3 - r) + 2v^2s = (v^3 - r)^2$, or $rv^3 + 4sv^2 - r^2 = 0$.

Ex. Following the same course as above, (or assuming the above result for $x$,) we shall have to put $3v^3 - 8v^2 - 9 = 0$, whence $v = 3$, and then we shall get $y^4 + 23y^2 - 140 = 0$, which gives $y$; and $x^2 + 3x = y$.

347. Let $P_m = A_0x^m + \&c$; and let $k_1$ be one of the values of $x$, less than $n$, which make $P_m$ divisible by $n$, with a quotient $Q_m$, suppose;

$$\therefore P_m - Q_m n = A_0 \{(x^m - k_1^m) + \frac{A_1}{A_0}(x^{m-1} - k_1^{m-1}) + \&c\} = A_0 (x - k_1) P_{m-1},$$

suppose. Now let $k_2$ be another value of $x$, less than $n$, which makes $P_m$ divisible by $n$, with quotient $Q_{m-1}$, suppose; then since the left-hand side of the above, upon putting $k_2$ for $x$, becomes div. by $n$, so also will the other side, $A_0 (x - k_1) P_{m-1}$. But $A_0$ is prime to $n$, and $k_1$ and $k_2$ are each *less* than $n$, so that $k_2 - k_1$ cannot be div. by $n$; $\therefore x = k_2$ must render $P_{m-1}$ divisible by $n$, or $P_m$ can only be made divisible by $n$ *once more* than $P_{m-1}$, viz. by putting $x = k_1$: and so on.

348. By Fermat's Theorem, $N^{n-1} - 1$ has $n - 1$ values of $N$, less than $n$, which make it a multiple of $n$; but

$$N^{n-1} - 1 = (N^{\frac{1}{2}(n-1)} + 1)(N^{\frac{1}{2}(n-1)} - 1),$$

and, by Ex. 347, neither of these factors can have more than $\frac{1}{2}(n-1)$ such values, and $\therefore$ each of them must have exactly so many values: so again,

$$N^{\frac{1}{2}(n-1)} - 1 = (N^{\frac{1}{4}(n-1)} + 1)(N^{\frac{1}{4}(n-1)} - 1),$$

and each of the factors $N^{\frac{1}{4}(n-1)} \pm 1$ must have $\frac{1}{4}(n-1)$ such values; &c.

349. By (175) $P$ for $p^m q^n = \dfrac{\lfloor m+n}{\lfloor m \lfloor n}$: now consider any one of these (as *ppqqqpppqp*); it is plain that $p^r$ may be introduced immediately *before* every $q$ and *after* the last, and so produce a different result; but the insertion of it elsewhere would only reproduce variations already formed; hence the number required $= (n+1)\dfrac{\lfloor m+n}{\lfloor m \lfloor n}$.

350. From Ex. 349 it appears that the chance of an event, whose prob. is $p$, happening altogether $m + r$ times, $r$ of them at least successively, and failing $n$ times, in $m + n + r$ trials, is

$$(n+1)\frac{\lfloor m+n}{\lfloor m \lfloor n} p^{m+r} q^n = p^r \{(n+1) C_m p^m q^n\},$$

where $C_m$ denotes the coeff. of $p^m q^n$ in $(p+q)^{m+n}$.

Now, in the case before us, $r = 6$, $m + n + r = 12$; $\therefore m + n = 6$, and since, in the twelve trials, tails may fall six, five, four, &c. times, or not at all, we must give to $n$ the different values 6, 5, 4, 3, 2, 1, 0; and thus we get the chance

$$= p^6 (7q^6 + 6.6pq^5 + 5.15p^2q^4 + 4.20p^3q^3 + 3.15p^4q^2 + 2.6p^5q + q^6) = \frac{1}{16},$$

since $p = q = \frac{1}{2}$.

351. As in Ex. 347, $P_m - Q_m n = A_0 (x - k_1) P_{m-1}$, or

$$P_m = Q_m n + A_0 (x - k_1) \{Q_{m-1} n + (x - k_2) P_{m-2}\}$$
$$= Q_m n + A_0 Q_{m-1} n (x - k_1) + A_0 (x - k_1)(x - k_2) \{Q_{m-2} n + (x - k_3) P_{m-3}\}$$
$$= Q_m n + A_0 Q_{m-1} n (x - k_1) + A_0 Q_{m-2} n (x - k_1)(x - k_2) + \&c.$$
$$+ A_0 Q_1 n (x - k_1) \dots (x - k_{m-1}) + A_0 (x - k_1) \dots (x - k_m):$$

but $P_m = A_0 x^m + A_1 x^{m-1} + A_2 x^{m-2} + \&c + A_{m-1} x + A_m$:

equating these values of $P_m$, (and observing that $A_0 (x - k_1) \dots (x - k_m) = A_0 x^m - A_0 S_1 x^{m-1} + A_0 S_2 x^{m-2} - \&c + (-1)^m S_m$,) we have

$$(A_0 S_1 + A_1) x^{m-1} - (A_0 S_2 - A_2) x^{m-2} + \&c \pm (A_0 S_m - (-1)^m A_m) =$$

an expression in $x$ of $m - 1$ dimensions, every term of which contains the factor $n$, so that it may be written

$$nB_1 x^{m-1} + nB_2 x^{m-2} + \&c + nB_m;$$

hence, equating coeff^s^, $A_0 S_1 + A_1$, &c, are each divisible by $n$.

352. By Fermat's Theorem, $x^{n-1} - 1$ has $n - 1$ values less than $n$, which make it divisible by $n$, viz. 1, 2, 3, &c $(n - 1)$; therefore, since $m$ in Ex. 351 = here $n - 1$, and is consequently even, $A_0 S_m - A_m = 1.2.3 \dots (n - 1) + 1$ is a multiple of $n$. Also $A_0 S_{m-1}$ is a multiple of $n$ (since $A_{m-1} = 0$), which

$$= 2.3 \dots (n - 1) + 1.3 \dots (n - 1) + 1.2.4 \dots (n - 1) + \&c.$$

$$= 1.2.3 \dots (n - 1) \{\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \&c + \frac{1}{n - 1}\}.$$

353. Putting $q = 1 - p$, the chance is, by Ex. 349,

$$(n + 1) \frac{\lfloor n}{n} p^r q^n + n \frac{\lfloor n}{\lfloor 1 \, \lfloor n - 1} p^{r+1} q^{n-1} + \&c,$$

or (writing backwards)

$$= p^{r+n} \{1 + 2n \frac{q}{p} + 3 \frac{n (n - 1)}{1.2} \frac{q^2}{p^2} + \&c\}$$

$$= p^{r+n} \{(1 + q)^n + n \frac{q}{p} (1 + q)^{n-1}\} = p^r (1 + nq).$$

354. $A$ must have left his end somewhere in the first $b+c$ minutes, and $B$ must have left his end somewhere in the first $a+c$ minutes. Let us find the chance that $B$ may have reached $A$'s end before $A$ started. Now $B$ could not have reached it before the end of the first $b$ minutes, and $A$ will have certainly left before the beginning of the last $a$ minutes; so that the event just supposed could have only happened by $B$ and $A$ being both at $A$'s end, somewhere in the $c$ minutes *between* the first $b$ and last $a$ minutes, the compound chance for which is $\frac{c}{a+c} \cdot \frac{c}{b+c}$; but still it is equally likely that $A$ may have started at any moment of this interval, and $\therefore$ before or after $B$ arrives; hence the chance of $B$'s reaching $A$'s end before $A$ leaves it is $\frac{c^2}{2(a+c)(b+c)}$ = also that of $A$'s reaching $B$'s end, before $B$ leaves it: $\therefore$ chance of their *not* meeting $= \frac{c^2}{(a+c)(b+c)}$, and of their meeting $= 1 - \frac{c^2}{(a+c)(b+c)}$.

# EQUATION PAPERS.

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1. 4. From (1), if $z = \sqrt{(x^2 - 15y - 14)}$, then $\frac{1}{5}z = \frac{1}{3}z^2 - 36 + \frac{14}{3}$:

from (2), $\frac{x^2}{16y^2} + \frac{x}{3y} = \frac{x}{2y}\sqrt{(\frac{x}{3y} + \frac{1}{4})} - \frac{1}{4}$,

$$\text{or } \frac{x^2}{16y^2} - \frac{x}{2y}\sqrt{(\frac{x}{3y} + \frac{1}{4})} + (\frac{x}{3y} + \frac{1}{4}) = 0,$$

a perfect square.

5. First gain = 500 × 18*d* – £20 = 350*s*. In second case, if $x$ = n° of gals of strong beer, then duty in pence

$$= 6x + \tfrac{6}{4}(500 - x) = \frac{3x + 500}{8} \text{ in shillings:}$$

$$\therefore \text{ second gain} = 500 \times 2s - \frac{3x + 500}{8} = \tfrac{10}{7} \text{ of } 350 \text{ by question.}$$

6. $x$ = n° of men at first; then n° of rations at first

$$= x + (x - 6) + \&c. \text{ to 8 terms} = 8x - 168;$$

and at end of 6th day they had used

$$x + (x - 6) + \&c. \text{ to 6 terms} = 6x - 90,$$

and had ∴ remaining $2x - 78 = 6 \times 61$ by question, whence $x = 222$. Now number of men alive at end of 6th day $= (x - 36) - 100 = 86$; and 6 terms of $86 + 76 + \&c. = 366$, the n° of rations remaining; ∴ they will last 6 days, and there will be then living 26 men.

7. Let $x : 1$ = ratio of weights of gold and alloy in the guinea,
$y : 1$ = ratio of values of equal weights of gold and alloy:

$$\text{then } \frac{x}{x+1} \times y + \frac{1}{x+1} \times 1 = \frac{24}{25}\{\frac{x}{x+1} \times \frac{249}{239}y + \frac{1}{x+1}\},$$

whence $xy = 239$: also by question

$$x^2 + y^2 = 11(x + y) + 233\tfrac{9}{121}.$$

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2. 4. From (1) by (85) $\frac{x + y}{\sqrt{(x^2 - y^2)}} = \sqrt{\frac{x + y}{x - y}} = \frac{9x + 17y}{9x + y}$,

whence (squaring, and by (85))

$$\frac{x}{y} = \frac{81x^2 + 162xy + 145y^2}{144xy + 144y^2}, \text{ whence } y = \tfrac{3}{5}x;$$

from (2) $(x^2+y)^2 - 2x(x^2+y) + x^2 = x^2 - x + y + 506$, or $z^2 = z + 506$.

5. Let $x =$ n° in army; then $\frac{x}{40} =$ n° in front, and $\therefore \frac{x}{10} =$ n° of spectators: $\therefore x + \frac{x}{10} = 45 \left(\frac{x}{40} - 100\right)$.

6. Let $2x =$ n° of persons, $y =$ payment of youngest;
$\therefore y + (y + 5) +$ &c. to $2x$ terms $= \{2y + (2x - 1) 5\} x = 345$ (1): also $\{2y + (x - 1) 5\} \times \frac{1}{2}x =$ sum of payments of younger half $= 22x$ (2).

7. Let $x =$ common ratio per hr, $y =$ n° of hrs after $A$ that $B$ starts: then, when $A$ passes either the geese or waggon, or when $A$ reaches L, $B$ is $xy$ miles distant: now rate of *approach* of $B$ and the geese is $x - \frac{3}{2}$ miles per hr; at which rate the distance $xy$ is diminished to zero, during the time in which the geese advance 5 miles:

$$\therefore xy : 5 :: x - \tfrac{3}{2} : \tfrac{3}{2} \quad (1).$$

Also rate of *approach* of $B$ and the waggon is $x + \frac{9}{4}$ miles per hr, at which rate the distance $xy$ is diminished to zero, while the waggon advances $(31 + \frac{2}{3}x) - (50 - 2x) = \frac{8}{3}x - 19$ miles.

$$\therefore xy : \tfrac{8}{3}x - 19 :: x + \tfrac{9}{4} : \tfrac{9}{4} \quad (2).$$

From (1) $xy = \frac{10x - 15}{3} = \frac{(8x - 57)(4x + 9)}{27}$ from (2), whence $x = 9$, $y = \frac{25}{9}$, $xy = 25$.

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8. 4. (1) cleared of fractions gives

$$y^2 - 81 = 2x^{\frac{3}{2}} (y + 9); \;\therefore y - 9 = 2x^{\frac{3}{2}} \text{ or } y + 9 = 0;$$

from (2) $y - 9 = x^{\frac{3}{2}} (\sqrt{y} - 3)$; $\therefore \sqrt{y} - 3 = 0$, or $\sqrt{y} + 3 = x^{\frac{3}{2}}$:

hence $\sqrt{y} - 3 = 0$, $y = 9$, $x = 0$; or $y - 9 = 2x^{\frac{3}{2}} = 2(\sqrt{y} + 3)$, whence $y = 25$, $x = 4$; &c.

5. Let $2x =$ distance required: then $x =$ rate of sailing from D to C: on return voyage, she makes $x$ miles at the rate $x - 6$, and $x$ at $x - 4$;

$$\therefore \frac{x}{x - 6} + \frac{x}{x - 4} = \text{time on return voyage} = \frac{6}{7} \times \frac{2x}{x - 6}.$$

6. $C$ the point of junction of two streets: $CA = x$, $CD =$ length of sewer $= y$; $AD = 6$ chains, $BD = y - 11$ chains; $DE$ the drain, $AE = 4$ chains: then (Euc. 6. 3) $\frac{BC}{AC} = \frac{BD}{AD}$, or

$$BC = x\frac{y-11}{6}, \text{ and } \frac{CD}{AD} = \frac{CE}{AE}, \text{ or } \frac{y}{6} = \frac{x-4}{4}: \quad (1)$$

$$\text{also } 9(AC+BC) = 9\left(x + x\frac{y-11}{6}\right) = 54 + xy. \quad (2)$$

7. Let $x$ = n° in family, $y$ = father's monthly savings in *shillings*, $z$ = common difference, $p$ = price of wheat per bushel at first: then $\{2y - (x-1)z\} \times \frac{1}{2}x$ = whole monthly savings

$$= \tfrac{1}{6}p\{12(y-8z)\} - 3\tfrac{1}{4} = 2p(y-8z) - 3\tfrac{1}{4}, \quad (1)$$

where $y - 8z$ = savings of 7th child in *shillings*.

$$\text{Also } (y-2z) + (y-6z) + 10 = 2p. \quad (2)$$

After the rise of wheat, their savings will only buy $2(y-8z)-2$ bushels; and $\therefore = 2(y-8z-1)(p+2)$, which (by question)

$$= \tfrac{1}{12}\{\text{former } annual \text{ saving} - 105s\}$$

$$= \text{former monthly saving} - 8\tfrac{3}{4} = 2p(y-8z) - 12. \quad (3)$$

Lastly, $(x-2)(y-2z-1)$ = whole monthly saving (after rise)

$$= 2p(y-8z) - 12. \quad (4)$$

From (2) $2y - 8z = 2p - 10$, from (3) $2(y-8z) = p - 4$, from which two equations we get $p = 8z + 6$, $y = 12z + 1$; putting these values in (4), we get $z = \frac{1}{32}(5x - 42)$;

$\therefore p = \frac{1}{4}(5x-18)$, $y = \frac{1}{8}(15x-118)$, $y - 8z = 4z + 1 = \frac{1}{8}(5x-34)$; and thus from (1) we get

$$5x^3 - 67x^2 - 54x + 2240 = 0,$$

one root of which is $x = 10$, which is easily found, since the integer values of $x$ must be factors of the last term, and the root is plainly (from the question) $> 9$.

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4. 2. From (2) $\sqrt{\frac{y-x}{a-x}} + 1 = \frac{5}{2}$, or $\sqrt{(a-x)} = \frac{2}{3}\sqrt{(y-x)}$;

$\therefore$ from (1) $\sqrt{y} = \sqrt{(y-x)}\,(1 + \frac{2}{3})$, and $\frac{y}{y-x} = \frac{25}{9}$, $y = \frac{25}{16}x$, which substitute in (1).

4. From (2) $\frac{1}{4}y^4 - y^2x + x^2 = x^2 - 1$, or $\frac{1}{2}y^2 = x + \sqrt{(x^2-1)}$; substitute for $y^2$ in (1); then

$$\frac{y}{2x} + \frac{2}{3}\cdot\frac{y - \sqrt{(x-1)}}{2x} = \frac{\sqrt{(x+1)}}{x}, \text{ or } 5y = 6\sqrt{(x+1)} + 2\sqrt{(x-1)};$$

square, and substitute again for $y^2$.

**5.** Let $x$ = price per bushel in *shillings*, $y$ = n° of bushels: then expected gain = $3y$ – int. on $xy$ for 6 months = $3y - \frac{1}{40}xy$: and, by question,

$$y + \tfrac{1}{40}xy = 5x \ (1), \quad 2y + \tfrac{1}{20}xy = (3y - \tfrac{1}{40}xy) - 10. \quad (2)$$

From (2) $\frac{1}{40}xy = \frac{1}{3}(y - 10)$ (3), by means of which (1) gives $y = \frac{5}{4}(3x + 2)$, which substitute for $y$ in (3).

6. Let $x$ = n° of days in length of voyage; then $x - 21$ = *expected* number of days, and the n° of daily allowances of water

$$=175(x-21) = \text{(by question) } 30 \times 175 + \{172 + 169 + \&\text{c. to } (x-30) \text{ terms}\}$$

$$= 5250 + \tfrac{1}{2}\{344 - 3(x - 31)\}(x - 30),$$

whence $x^2 - 59x - 1580 = 0$.

7. Let $x$ = horary influx at leak, $y$ = n° of gals in hold at first, $z$ = n° of gals $A$ pumps per hour: then $\frac{2}{3} \times \frac{5}{4} \times z = \frac{5}{6}z$ = n° $B$ pumps. Now $z - x$ = rate at which $A$ *empties* the hold; $\therefore \dfrac{y}{z - x}$ = time during which $B$ pumps; and $\frac{5}{6}z - x$ = rate at which $B$ *empties*; $\therefore (\frac{5}{6}z - x) \times \dfrac{y}{z - x}$ = n° of gals pumped by $B$, and n° remaining in hold $= y\left\{1 - \dfrac{\frac{5}{6}z - x}{z - x}\right\} = \dfrac{yz}{6(z - x)}$, which $A$ (*his* rate of emptying being $z - x$ gals per hour) will pump out in $\dfrac{7z}{6(z - x)^2}$ hours.

$$\text{Hence } \frac{y}{z - x} + \frac{yz}{6(z - x)^2} = 13\tfrac{1}{3}, \text{ whence } y = \frac{80(z - x)^2}{7z - 6x}; \quad (1)$$

$$\text{and } \frac{y}{(z + \frac{5}{6}z) - x} = 3\tfrac{3}{4}, \text{ whence } y = \tfrac{5}{8}(11z - 6x); \quad (2)$$

$$\text{also } 3\tfrac{3}{4} \times (z) = \frac{yz}{6(z - x)^2} \times (z) + 100. \quad (3)$$

From (1) and (2), equating values of $y$, we get $z = 2x$; and then from (2) $y = 10x$; and then from (3) $x = 120$ gals.

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**5.** 3. Mult. by $x^{\frac{2}{3}}$; then $x^3 + 41 = 97 + x^{\frac{3}{2}}$, or $x^3 - x^{\frac{3}{2}} - 56 = 0$.

4. From (1) $(x^2y^2 - 272) + 2y\sqrt{(x^2y^2 - 272)} + y^2 = 81y^2$, whence $\sqrt{(x^2y^2 - 272)} = 8y$ or $-10y$:

from (2) $x^2y^2 - 35xy - 36 = 0$, whence $xy = 36$ or $-1$.

5. Let $x = A$'s age by Jan. 1, 1799; then
$\{x+(x+1)+\&c$ to 8 terms$\}\times 4d=(2x+7)\,4\times 4d=$(by question) $476\times 4d$;

$$\therefore\ 2x+7=119.$$

6. Let $x =$ price of each *canal* share: then $5x+595 =$ profits of *all*, and $\frac{1}{15}(5x+595)=\frac{1}{3}(x+119) =$ profits of *each*: $\therefore\ \frac{1}{3}(x+119)-173 =$ price of *steam-boat* share, and $A$'s loss $=\{\frac{1}{3}(x+119)-173\}+\frac{1}{3}(x+119)+\frac{1}{8}$ of $368=419$.

7. Let $x =$ width of $A$'s and $B$'s larger, $y$ of their smaller buildings: then $x^3+y^3 =$ content of $A$'s, $x^3+x^2y =$ content of $B$'s, and $x^2y-y^3=y(x^2-y^2)=73728$: (1)
also $x^2-y^2 =$ area of base of $C$'s $=2688-8\sqrt{(x^2-y^2)}$. (2)
From (2) $\sqrt{(x^2-y^2)}=48 =$ side of $C$'s; from (1) $y=32$.

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6. 3. Divide every term by $x^{\frac{1}{m}}$; then

$$a^2b^2x^{\frac{m-n}{mn}}-4(ab)^{\frac{3}{2}}x^{\frac{m-n}{2mn}}-(a-b)^2=0,$$

$$\text{whence } x^{\frac{m-n}{2mn}}=\pm\left(\frac{1}{\sqrt{a}}\pm\frac{1}{\sqrt{b}}\right)^2,\ \text{ and } x=(a^{-\frac{1}{2}}\pm b^{-\frac{1}{2}})^{\frac{4mn}{m-n}}.$$

4. From (1) $x^4-2x^2y^2+y^4=1+2xy+x^2y^2$;

$$\therefore\ x^2-y^2=1+xy,\ \text{or } x^2-xy=y^2+1;\quad (a)$$

$\therefore$ (2) $(x+y)(x^2-xy+y^2)=(x+y)(2y^2+1)=(2y^2+1)(x+1)$, whence $2y^2+1=0$, and also $x+y=x+1$, or $y=1$, whence $(a)$ $x^2-x=2$.

5. Let $x =$ rate backwards, $4x =$ rate forwards; then, calling the distance *unity* (93 Ex. 5), we have calculated time

$$=\frac{\frac{3}{4}}{4x}+\frac{\frac{1}{4}}{x}=\frac{7}{16x},\ = \text{also } \frac{\frac{3}{4}}{4x+2}+\frac{\frac{1}{4}}{x-\frac{1}{5}},\ \text{by question.}$$

6. Let $100x =$ debt, $r =$ rate of int. per cent; $y =$ price *per cent* of bonds at first; then $\frac{1}{5}rx =$ n° of cents of Spanish bonds, and their produce, when sold at the end of the year, (their int. having paid also the int. of the debt,)

$$=\frac{1}{5}rx(y-40)=100x-400;\quad (1)$$

$$\text{also}\quad \frac{1}{5}rx(y)=100x+250;\quad (2)$$

$\frac{1}{5}rx(50)=$ (debt + half-year's Int.) $-300=100x+\frac{1}{2}rx-300$. (3)
From (1) and (2) $8rx=650$; from (3) $\frac{1}{5}\times\frac{650}{8}\times 50=100x+\frac{325}{8}-300$
whence $100x=\frac{8575}{8}=$ £1071 17*s* 6*d*.

7. Let $x$ = n° of convicts at first: then $(p-1)\,nax$ = expected gain: but actual expense of each man for $n$ weeks

$$= a\,\{(1+r)+(1+r+r^2)+\&c\} = a\,\frac{(1-r)+(1-r^2)+(1-r^3)+\&c}{1-r}$$

$$= a\,\frac{n - r\,\dfrac{1-r^n}{1-r}}{1-r} = \frac{a}{(1-r)^2}\,\{n\,(1-r) - r\,(1-r^n)\}.$$

Also that of fresh convicts (since $c$ of them will have to be fed for $(n-1)$ weeks, $c$ for $(n-2)$ weeks, &c, that is, $c$ will require the food for the $(n-1)$th week, $2c$ for the $(n-2)$th week, &c,) will be

$$ac\,\{\frac{1-r^n}{1-r} + 2\,\frac{1-r^{n-1}}{1-r} + \&c\} = \frac{ac}{1-r}\,\{(1+2+\&c) - (r^n + 2r^{n-1} + \&c)\}$$

$$= \frac{ac}{1-r}\,[\tfrac{1}{2}n\,(n-1) - \{(n-1)\,\frac{r}{1-r} - \frac{r^2\,(1-r^{n-1})}{(1-r)^2}\}] \quad [184]$$

$$= \frac{ac}{(1-r)^3}\cdot\{\tfrac{1}{2}n\,(n-1)\,(1-r)^2 - r\,\{n\,(1-r) - (1-r^n)\}\};$$

and their produce $= pac\,(1+2+3+\&c) = \frac{1}{2}n\,(n-1)\,pac$:
∴ (by question)

$$(p-1)\,nax = pnax + \tfrac{1}{2}n\,(n-1)\,pac - \frac{ax}{(1-r)^2}\,\{n\,(1-r) - r\,(1-r^n)\}$$

$$- \frac{ac}{(1-r)^3}\,\{\tfrac{1}{2}n\,(n-1)\,(1-r)^2 - r\,\{n\,(1-r) - (1-r^n)\}\};$$

$$\therefore \frac{ax}{(1-r)^2}\,\{n\,(1-r) - r\,(1-r^n) - n\,(1-r)^2\} \text{ or } \frac{rax}{(1-r)^2}\,\{n\,(1-r) - (1-r^n)\}$$

$$= \frac{ac}{(1-r)^3}\,\{\tfrac{1}{2}pn\,(n-1)\,(1-r)^3 - \tfrac{1}{2}n\,(n-1)\,(1-r)^2 + r\,\{n\,(1-r) - (1-r^n)\}\};$$

$$\therefore x = \frac{c}{1-r}\left\{1 + \frac{n\,(n-1)\,(1-r)^2\,\{p\,(r-1)-1\}}{2r\,\{n\,(1-r) - (1-r^n)\}}\right\}.$$

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7. 3. Multiply by 50: then

$$\frac{480 - 75\sqrt{x} - 50x}{5\sqrt{x} - 8} + 31 = \frac{232 + 80\sqrt{x} - 50x}{5\sqrt{x} - 8} = \frac{592\sqrt{x} - 40x^{\frac{3}{2}}}{4x - 7},$$

$$\text{or } -10\sqrt{x} + \frac{232}{5\sqrt{x}-8} = \sqrt{x}\,(-10 + \frac{522}{4x-7}), \text{ whence } \frac{4}{5\sqrt{x}-8} = \frac{9\sqrt{x}}{4x-7}.$$

4. From (1), $\left\{(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}) - \frac{x}{2}\right\}^2 = \frac{x^2}{4} - x\,\sqrt{\frac{y}{2}} + \frac{y}{2} = \left\{\frac{x}{2} - \sqrt{\frac{y}{2}}\right\}^2,$

$$\text{whence } \sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}} = x - \sqrt{\frac{y}{2}} \text{ (3), or } = \sqrt{\frac{y}{2}} \text{ (4):}$$

From (2), (mult. by $2\sqrt{(xy)}$) we get

$$18x + 6y = \{21\sqrt{(2x)} - 1\}\,y + 1,$$

$$\text{or } \{21\sqrt{(2x)} - 7\}\,y = 7\,\{3\sqrt{(2x)} - 1\}\,y = 18x - 1$$

$$= \{3\sqrt{(2x)} + 1\}\,\{3\,\{\sqrt{(2x)} - 1\};$$

$$\therefore\ 3\sqrt{(2x)} - 1 = 0, \quad x = \tfrac{1}{18}:$$

to find $y$, substitute for $x$ in (3) and (4).

Also $7y = 3\sqrt{(2x)} + 1$, which, put for $y$ in (4), gives values of $x$: but (3) would become complicated by this substitution.

5. Let $x$ = n° of bags purchased; $\therefore\ x + 18$ = n° intended, and $\frac{8}{5}(x + 18)$ = money sent in £'s, and $\dfrac{8}{5}\cdot\dfrac{x+18}{x}$ = price given per bag: $\therefore$ (by question)

$$\{5\tfrac{1}{4} + \tfrac{1}{8}(x + 18)\}\,\{\frac{8}{5}\cdot\frac{x+18}{x} - \frac{8}{5}\} = 10\tfrac{7}{20}.$$

6. Let $x$ and $y$ be the rates per hour of A and B, $z$ the distance $CD$: then A had gone $z - 20$ miles, and B 20 miles when they first met;

$$\therefore\ \frac{z-20}{x} = 3 + \frac{20}{y}\ \ (1), \quad\text{and}\quad \frac{20}{x} = \frac{z-20}{y} - 1\ \ (2):$$

also A had gone $\frac{6}{7}z$, and B $\frac{1}{7}z$, when they next met;

$$\therefore\ \frac{6z}{7x} = \frac{z}{7y} + 3 + \frac{28}{y}\ \ (3).$$

From (1), $\dfrac{z}{x} = 3 + 20\,(\dfrac{1}{x} + \dfrac{1}{y})$ (4), from (2), $\dfrac{z}{y} = 1 + 20\,(\dfrac{1}{x} + \dfrac{1}{y})$ (5);

$$\therefore\ \frac{z}{x} = 2 + \frac{z}{y},\quad\text{and (3)}\quad \frac{6}{7}\,(2 + \frac{z}{y}) = \frac{z}{7y} + 3 + \frac{28}{y},$$

whence $\dfrac{z}{y} = \dfrac{9}{5} + \dfrac{196}{5y} = 1 + \dfrac{20}{x} + \dfrac{20}{y}$ from (5); $\therefore\ \dfrac{y}{x} = \dfrac{y+24}{25}$;

$$\text{and } \frac{(4)}{(5)} \text{ gives } \frac{y}{x} = \frac{3 + 20\,(\frac{1}{x} + \frac{1}{y})}{1 + 20\,(\frac{1}{x} + \frac{1}{y})} = \frac{3y + 20\,\frac{y}{x} + 20}{y + 20\,\frac{y}{x} + 20},$$

where substituting for $\dfrac{y}{x}$ we get finally $9y^2 - 63y - 196 = 0$ where $y = \frac{28}{3}$.

7. Let $x$ = lowest debt, $y$ = common diff., $n$ = n° of creditors: then sum of debts $= \{2x + (n - 1)\,y\}\,\frac{1}{2}n$, and $\therefore$ assets

$$= \tfrac{1}{40}ny\,\{2x + (n - 1)\,y\} = 2x + (n + 1)\,y;\ \ (1)$$

also, when the highest claim is removed, sum of debts

$$=\{2x+(n-2)\,y\}\,\tfrac{1}{2}\,(n-1);$$

$$\therefore\ \text{dividend}=\frac{\text{assets}}{\text{debts}}=\frac{2x+(n+1)\,y}{\{2x+(n-2)\,y\}\,\frac{1}{2}(n-1)}=\frac{y+\frac{8}{3}}{20}=\frac{3y+8}{60};\ (2)$$

and $\frac{8}{3}\{2x+(n-2)y\}=192$ (3), whence $2x+(n-2)\,y=72$;

$\therefore$ (1) gives $\frac{1}{40}ny\,(y+72)=3y+72$, or $\quad n=\dfrac{120\,(y+24)}{y\,(y+72)}$,

and (2) gives $\dfrac{3y+72}{72\times\frac{1}{2}(n-1)}=\dfrac{3y+8}{60}$, or $n-1=\dfrac{5\,(y+24)}{3y+8}$;

$$\therefore\ 1=\frac{120\,(y+24)}{y\,(y+72)}-\frac{5\,(y+24)}{3y+8},$$

whence $y^3+43y^2=48y+2880$, or $y^3-2y-48y=2880-45y^2$,
that is, $(y+6)\,(y-8)=45\,(64-y^2)$, whence $y=8$.

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**8.** 2. From (1) $\sqrt[3]{(x+y)}\,(\frac{1}{y}+\frac{1}{x})=\dfrac{(x+y)^{\frac{4}{3}}}{xy}=\dfrac{64}{63}$, from (2) $\dfrac{(x-y)^{\frac{4}{3}}}{xy}=\dfrac{4}{63}$;

$$\therefore\ \left(\frac{x+y}{x-y}\right)^{\frac{4}{3}}=16,\quad \frac{x+y}{x-y}=8,\quad \frac{y}{x}=\frac{7}{9},\ \text{\&c.}$$

3. Divide each term by $x^{\frac{1}{p}}$; then

$$x^{\frac{p-q}{2pq}}=\frac{a^2-b^2}{2\,(a^2+b^2)}\,(1+x^{\frac{p-q}{pq}}),\ \text{ or }\ z\doteq\frac{a^2-b^2}{2\,(a^2+b^2)}\,(1+z^2),\ \text{suppose,}$$

$$\text{whence }\ z=\frac{a+b}{a-b}\ \text{ or }\ \frac{a-b}{a+b}.$$

4. From (1) $(\frac{5}{4}x^2-2y+8)+2x\sqrt{(\frac{5}{4}x^2-2y+8)}+x^2=\frac{9}{4}x^2+6x+4$;
$\therefore\ \sqrt{(\frac{5}{4}x^2-2y+8)}=-x+(\frac{3}{2}x+2)=2+\frac{1}{2}x$, whence $y=\frac{1}{2}x^2-x+2$: (3)
from (2) $(x+y)-(\frac{3}{2}x^2-3y)\sqrt{(x+y)}=4x^2-6x$;
$\therefore$ by (3) $(x+y)-(3x-6)\sqrt{(x+y)}+(\frac{3}{2}x-3)^2=\frac{25}{4}x^2-15x+9$;
$\therefore\ \sqrt{(x+y)}=(\frac{3}{2}x-3)\pm(\frac{5}{2}x-3)=4x-6$ or $-x$,
and $y=16x^2-49x+36$, (4) or $y=x^2-x$; (5)
obtain the roots by comparing (3) and (4), or (3) and (5).

5. Let $x$ = exp., of which $\frac{1}{6}x$ = taxes; then 1st year's produce $=x+30$; but 2nd year's rent = £40, taxes $=\frac{1}{12}x$, produce $=\frac{4}{3}(x+30)$, and other exp. = (as before) $\frac{5}{6}x$;

$$\therefore\ \tfrac{4}{3}\,(x+30)=(\tfrac{5}{6}x+\tfrac{1}{12}x+40)+20+5.$$

6. Let $2x$ = whole distance, $y$ = B's rate, $z$ = C's: then, when A and B meet, they will have done between them the distance $x$ at a compound rate $10+y$; $\therefore$ time $=\frac{x}{y+10}$, and dist. from N $=\frac{10x}{y+10}$: hence time at which C overtakes B $=\frac{x}{y+10}+4$; and $\therefore$, since $x$ = dist. between B and C at first, which is diminished at the rate $z-y$, we have

$$\left(\frac{x}{y+10}+4\right)(z-y)=x. \quad (1)$$

Again, dist. from N at which A and B meet on return $=\frac{10x}{y+10}$: $\therefore$ (leaving out the *common* time of resting) A had gone $2x+\left(2x-\frac{10x}{y+10}\right)$ miles while B had gone $x+\frac{10x}{y+10}$;

$$\therefore 4x-\frac{10x}{y+10} : x+\frac{10x}{y+10} :: 10 : y, \quad (2)$$

and time spent in this *travelling* $=\frac{4x}{10}-\frac{x}{y+10}$; $\therefore$ time spent in travelling, when A and C meet, is $\frac{4x}{10}-\frac{x}{y+10}-1$, in which time they will have gone together $6x$ miles at rate $10+z$;

$$\therefore \left(\frac{4x}{10}-\frac{x}{y+10}-1\right)(10+z)=6x. \quad (3)$$

From (2) $5x : \frac{50x}{y+10} :: 10+y : 4y-10$, whence $y=5$: from (1) $z-5=\frac{15x}{x+60}$, from (3) $z+10=\frac{18x}{x-3}$.

7. Let $4x$, $21x$ be quantities of fluid at first in $P$ and $Q$: $p:1$ the ratio of $W$ to $S$ in $P$, $q:1$ in $Q$:

$$\therefore \frac{4px}{p+1}=W \text{ in } P,\ \frac{4x}{p+1}=S \text{ in } P,\ \frac{21qx}{q+1}=W \text{ in } Q,\ \frac{21x}{q+1}=S \text{ in } Q;$$

$$\therefore \text{strength of } P=\left(\frac{4px}{p+1}\times 1+\frac{4x}{p+1}\times 3\right)\div 4x=\frac{p+3}{p+1}=a,$$

$$\text{strength of } Q=\frac{q+3}{q+1}=b:$$

Let $4xy$ = quantity pumped by A; then $3x(1-y)$ = do. by B; $\therefore$ quantity now in $Q = x(y+24)$, and strength of $Q$

$$= \frac{21xb + x(y+3)a}{x(y+24)} = \frac{a(y+3)+21b}{y+24} = \frac{12}{13}b: \quad (1)$$

on second supposition, when A stopped, $P$ contains $4x(1-y)$ and $Q$ $x(21+4y)$, the latter of strength $\frac{21b+4ay}{21+4y} = c$; hence, if B pumps out of $Q$ into $P$ as much as before, namely, $3x(1-y)$, $P$ will contain $7x(1-y)$ and be of strength

$$\frac{4(1-y)a + 3(1-y)c}{7(1-y)} = \frac{4a+3c}{7} = \sqrt{(ab)}. \quad (2)$$

Again, ratio of $W$ to $S$ in mixture now pumped by $B$ = ratio of $S$ to $W$ in former case, since he pumped the same quantity in each case. Now, generally, if $s$ be the strength of a mixture consisting of $u$ parts of wine and $v$ of spirits, we have $s = \frac{u+3v}{u+v}$, or $\frac{u}{v} = \frac{s-3}{1-s}$:

$$\text{hence } \frac{c-3}{1-c} = \frac{1-a}{a-3}, \quad \text{or } a + c = 4. \quad (3)$$

Also $c = \frac{21b+4ay}{21+4y}$, whence

$$y = \frac{21}{4} \cdot \frac{b-c}{c-a} = \frac{21}{8} \cdot \frac{a+b-4}{2-a} = \text{also } \frac{3(13a-5b)}{12b-13a} \text{ from (1);} \quad (4)$$

and (2) with (3) gives $a + 12 = 7\sqrt{(ab)}$, whence, putting $b = m^2a$, we get $a = \frac{12}{7m-1}$:

substituting the value for $a$ in (4), we get (upon reduction)

$$4(13-5m^2)(m-1) = (3m^2-7m+4)(12m^2-13)$$
$$= (3m-4)(m-1)(12m^2-13);$$

whence $m = 1$, and $52 - 20m^2 = 36m^3 - 48m^2 - 39m + 52$, whence $m = 0$, or $\frac{3}{2}$, or $-\frac{13}{18}$, of which, taking $m = \frac{3}{2}$, we have $y = \frac{3(13-5m^2)}{12m^2-13} = \frac{3}{8}$; $\therefore$ the ratio required or $\frac{4y}{3(1-y)} = \frac{4}{5}$.

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**9.** 3. Expanding $(x^{\frac{1}{2}}-1)^2$ and reducing, we have

$$(a^{4m}-1)x - 2(a^{4m}+1)x^{\frac{1}{2}} + (a^{4m}-1) = 0,$$

$$\text{whence } x^{\frac{1}{2}} = \frac{(a^{4m}+1) \pm 2a^{2m}}{a^{4m}-1} = \frac{a^{2m} \pm 1}{a^{2m} \mp 1}.$$

Better to assume $\left.\begin{array}{l} x = v + u \\ y = v - u \end{array}\right\}$ very elegant

4. From (2),

$$9 = \left\{\left(\frac{x+y}{2}\right)^{\frac{1}{2}} + \left(\frac{x-y}{2}\right)^{\frac{1}{2}}\right\} \left\{\frac{x+y}{2} - \left(\frac{x^2-y^2}{4}\right)^{\frac{1}{2}} + \frac{x-y}{2}\right\}$$

$$= \text{by (1) } \frac{y}{14}\{2x+\sqrt{(x^2-y^2)}\}\{\frac{2x-\sqrt{(x^2-y^2)}}{2}\} = \frac{y}{28}(3x^2+y^2);$$

$$\therefore 3x^2y + y^3 = 252, \quad (3)$$

and, by squaring (2), $x^3 + 3xy^2 + (x^2-y^2)^{\frac{3}{2}} = 324$: (4)

$$(4) + (3) \text{ gives } (x+y)^3 = 576 - (x^2-y^2)^{\frac{3}{2}},$$

$$(4) - (3) \text{ gives } (x-y)^3 = 72 - (x^2-y^2)^{\frac{3}{2}};$$

$$\therefore (x^2-y^2)^3 = 41472 - 648\,(x^2-y^2)^{\frac{3}{2}} + (x^2-y^2)^3,$$

whence $(x^2-y^2)^{\frac{3}{2}} = 64$, and $\therefore (x+y)^3 = 512$, $(x-y)^3 = 8$.

5. Let $x$ = drain on B; $\therefore 3x$ = his capital; $y$ = drain at first on A; $\therefore 3y + 2(3y)$ or $9y$ = his capital:

hence $3x + 9y = 7(x+y)$, (1), and $4x = 4000$. (2)

6. Let $x$ = charge *per annum* for large burner, $y$ for small one:

then $5x + 3y = x + 8y$: (1)

and $\frac{1}{6}(x + 8y)$ = additional charge on these for extra time, since they burn now $\frac{7}{6}$ of their former time; also $\frac{7}{6}y$ = annual charge on each of the additional burners;

$$\therefore \tfrac{1}{6}(x+8y) + \tfrac{7}{6}(2y) = 31, \quad (2)$$

and whole annual charge $= \frac{7}{6}(x + 10y)$.

7. Let $2x$ = distance required, $y$ = rate at first, $z$ hrs = gain of watch per hour; then, by question,

$$\frac{x}{y} + \frac{x}{\frac{4}{3}y} = 8\tfrac{3}{4}, \quad (1)$$

whence $\frac{x}{y} = 5$, and $\therefore$ 5 hrs = time of half distance.

Now in 5 hrs the watch gains $5z$ hrs, and he thinks he has only $4 - 5z$ hrs remaining to go $x$ miles, which he expects to do at his increased rate, $\frac{4}{3}y$;

$$\therefore x = (4 - 5z) \times \tfrac{4}{3}y. \quad (2)$$

On second supposition, at end of 14th mile, he will have taken *really* $\frac{14}{y}$ hrs, *apparently* $\frac{14}{y}(1-z)$ hrs; $\therefore$ by question,

$$9 - \frac{14}{y}(1-z) = (9 - \frac{14}{y}) + \frac{7}{60}.$$

From (3) $y = 120z$, from (1) and (2); $y = 6$; $\therefore 2x = 60$.

**10.** 3. $\dfrac{243+324\sqrt{(3x)}}{16x-3} = \dfrac{81\sqrt{3}\,(4\sqrt{x}+\sqrt{3})}{(4\sqrt{x}+\sqrt{3})\,(4\sqrt{x}-\sqrt{3})} = 16x+3-8\sqrt{(3x)}$
$= (4\sqrt{x}-\sqrt{3})^2$:

$\therefore 81\sqrt{3} = (4\sqrt{x}-\sqrt{3})^3$, or $3\sqrt{3} = 4\sqrt{x}-\sqrt{3}$, whence $x=3$; and the factor $4\sqrt{x}+\sqrt{3}=0$ gives $x=\frac{3}{16}$.

4. From (1) $(3y^2-11+2x)+2\sqrt{(3y^2-11+2x)}+1 = y^2+4y+4$;
$\therefore \sqrt{(3y^2-11+2x)} = -1+y+2 = y+1$, whence $x = 6+y-y^2$:

$$\therefore \text{(2) gives } \sqrt{(y^2+2y+1)} \text{ or } y+1 = \frac{6+2y-y^2}{6-y^2},$$

whence $y=0$ or $\pm 2$.

5. Let $x$, $y$ be $\text{n}^{\text{os}}$ in bags, $z = \text{n}^\circ$ in handful:

$$\text{then } x-z = (y-z)^3 = z^2, \text{ and } y-z = z^{\frac{2}{3}}:$$

also $(y-z)^2 = \text{n}^\circ$ left in the larger after second drawing;

$$\therefore (y-z)^2+(y-z) = \tfrac{5}{3}y, \text{ or } z^{\frac{4}{3}}+z^{\frac{2}{3}} = \tfrac{5}{3}(z+z^{\frac{2}{3}}), \text{ whence}$$

$3z^{\frac{2}{3}}-5z^{\frac{1}{3}}-2=0$ and $z^{\frac{1}{3}}=2$, or $z=8$, $y=z+z^{\frac{2}{3}}=12$, $x=z+z^2=72$.

6. Let $x$=dist. from A to B, $y$=waterman's rate without stream;

$$\therefore \frac{x}{y+4}+\frac{x}{y-4} = \frac{2x}{y}+\frac{39}{60}, \text{ (1)} \quad \frac{x}{\frac{3}{2}y+4}+\frac{x}{\frac{3}{2}y-4} = \frac{2x}{\frac{3}{2}y}+\frac{8}{60}: \text{ (2)}$$

$$\text{from (1) } \frac{32x}{y(y^2-16)} = \frac{13}{20}, \quad \text{from (2) } \frac{32x}{y(9y^2-64)} = \frac{1}{20};$$

$$\therefore \frac{9y^2-64}{y^2-16} = 13, \text{ and } y=6.$$

7. Let $x = \text{n}^\circ$ of yds done by A, $y$ by B; $a = \text{n}^\circ$ of yds done by A in a month, $b$ by B: then $\dfrac{x}{a}$ = time taken by A, $\dfrac{y}{b}$ by $B$, and $\dfrac{x+y}{a+b}$ = time they would have taken together for it:

$$\therefore \frac{b}{a+b}(x+y) = 3a, \quad \frac{a}{a+b}(x+y) = 12b = x+36;$$

$$\text{hence } 3a^2 = 12b^2, \text{ or } a = 2b:$$

$$\therefore x+y = \frac{3a(a+b)}{b} = 9a, \text{ and } x+36 = 12b = 6a;$$

$\therefore y-36 = 3a$, and A's time or $\dfrac{x}{a} = \dfrac{6a-36}{a}$, B's time or

$\frac{y}{b} = \frac{6a + 72}{a}$, of which the latter is greater; therefore, by first part of question,

$$\frac{4}{5} \cdot \frac{6a + 72}{a} = \frac{x + y}{a + b} = \frac{9a}{\frac{3}{2}a} = 6,$$

whence $a = 48$, and $x + y = 9a = 432$.

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11. 2. From (1) $c(ay + bx) = xy(x + y)$; from (2) $xy\{c(ay + bx) - cxy\}$

$$= xy\{xy(x + y) - cxy\} = x^2y^2(x + y - c) = abc^2(x + y - c);$$

$$\therefore x^2y^2 = abc^2, \text{ (3) } \quad \text{or } x + y = c: \text{ (4)}$$

from (1) and (3) $ay + bx = \sqrt{(ab)}\,(x + y)$, or $y\sqrt{a} = x\sqrt{b}$;

$$\therefore \text{(3) } x = \pm\sqrt{(ac)}, \quad y = \pm\sqrt{(bc)}:$$

from (1) and (4) $ay + bx = xy$, whence (4) $x + \frac{bx}{x - a} = c$.

3. $(x^2 + 2)^2 + \frac{3}{16} = (2x^2 + 3)\sqrt{(x^2 + 2)} = 2(x^2 + 2)^{\frac{3}{2}} - \sqrt{(x^2 + 2)}$:
$\therefore (x^2 + 2)^2 - 2(x^2 + 2)^{\frac{3}{2}} + (x^2 + 2) = (x^2 + 2) - \sqrt{(x^2 + 2)} - \frac{3}{16}$,
or $z^2 - z + \frac{3}{16} = 0$, if $z = (x^2 + 2) - \sqrt{(x^2 + 2)}$.

4. From (2) $(x^{\frac{2}{3}} + y^{\frac{2}{3}})\{3x^{\frac{2}{3}}y^{\frac{2}{3}} + \frac{91}{216}\sqrt{(x^{\frac{2}{3}} + y^{\frac{2}{3}})}\} = \frac{125}{216} - x^2 - y^2$,

$$\text{or } (x^2 + 3x^{\frac{4}{3}}y^{\frac{2}{3}} + 3x^{\frac{2}{3}}y^{\frac{4}{3}} + y^2) + \frac{91}{216}(x^{\frac{2}{3}} + y^{\frac{2}{3}})^{\frac{3}{2}} = \frac{125}{216},$$

$$\text{or } 216z^2 + 91z - 125 = 0, \quad \text{if } z = (x^{\frac{2}{3}} + y^{\frac{2}{3}})^{\frac{3}{2}},$$

$$\text{whence } x^{\frac{2}{3}} + y^{\frac{2}{3}} = \frac{25}{36} \text{ or } 1:$$

using the first of these in (1), we get

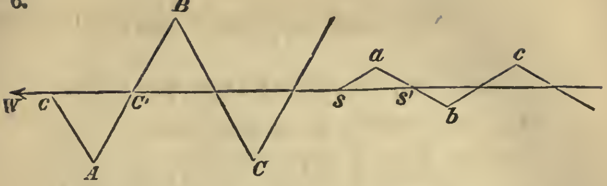
$$0 = 48xy - 241x^{\frac{2}{3}}y^{\frac{2}{3}} + 25 = 48x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{1}{3}}y^{\frac{1}{3}} - 5) - (x^{\frac{2}{3}}y^{\frac{2}{3}} - 25)$$

$$= (x^{\frac{1}{3}}y^{\frac{1}{3}} - 5)(48x^{\frac{2}{3}}y^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} - 5); \quad \therefore x^{\frac{1}{3}}y^{\frac{1}{3}} = 5, \tfrac{1}{3}, \text{ or } -\tfrac{5}{16}:$$

with $x^{\frac{1}{3}}y^{\frac{1}{3}} = \frac{1}{3}$, we have $x^{\frac{2}{3}} \pm 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = \frac{25}{36} \pm \frac{2}{3}$, or $x^{\frac{1}{3}} \pm y^{\frac{1}{3}} =$ &c.

5. Let $x$ be the hour, and $\therefore (x - 1) =$ n° of seconds taken by the fastest clock, *A*, in striking: then $3(x - 1) - 2 =$ n° of seconds *B* has been striking, and since the last stroke of *A* coincides with one of *B*'s, $3(x - 1) - 2$ or $3x - 5$ must be a multiple of 4; $\therefore x$ may be 3, 7, or 11. Now, if *all* the strokes had been heard, the n° heard would have been $2x$; and, consequently, since the n° actually heard is 19, it is plain that $x$ cannot be 3 or 7, but must be 11.

6.



On consideration, it will be found that the above figure represents the problem, $C$ and $S$ being the positions of Cutter and Smuggler at starting, $CW$ the direction of wind, $\angle ACC' = 5\frac{1}{3}$ points $= 60°$, (since the 4 right angles are divided on the compass in 32 *points*,) $\angle aSS'$, on the *other* tack, $= 2\frac{2}{3}$ points $= 30°$, $CA$, $Sa$ the first *half* tacks, $AB$, $BC$, &c, $ab$, $bc$, &c, the successive tacks: then since $AB : ab :: 4p : \frac{1}{3}p\sqrt{3} :: 4\sqrt{3} : 1 ::$ rate of $S$ : rate of $C$, it appears that they will begin and end each tack together, and also be in the line of the wind's action, $SCW$, together. Hence if $n$ be the number of tacks required, we shall have distance between the vessels =

$$3q - n(4p \cos 60° - \tfrac{1}{3}p\sqrt{3}.\cos 30°) = 3q - \tfrac{3}{2}np \overset{=}{<} r;\ \text{whence } n.$$

7. Let $x$ = length of levels at first irruption, $y$ = do. at second; $z$ = breadth = height: then $2z$ = side of shaft, and $5z$ = height, $2xz + 4z^2 = 2z(x + 2z)$ = area of horizontal section of whole tunnel at first irruption, $2z(y + 2z)$ = ditto at second, $2xz^2 + 20z^3 = 2z^2(x + 10z)$ = vol. of tunnel at first irruption, $2z^2(y + 10z)$ at second: let $u$, $v$ be the volumes of water in cubic feet, entering *per* minute in first and second irruptions: then rate at which it rose at first in the shaft (that is, after the levels were filled) was

$$\frac{u}{4z^2} = \frac{8v}{2z^2(y + 2z)}, \quad \text{or} \quad \frac{u}{v} = \frac{16z}{y + 2z}. \quad (1)$$

Also the three rates,

$$\frac{u}{2z(x + 2z)}, \quad \frac{v}{2z(y + 2z)}, \quad \frac{v}{2z(y + 110 + 2z)},$$

form an A.P. whose common difference is $\dfrac{v - u}{36z^2}$;

$$\therefore \frac{u}{2z(x+2z)} - \frac{v}{2z(y+2z)} = \frac{v-u}{36z^2}, \quad (2)$$

$$\frac{v}{2z(y+2z)} - \frac{v}{2z(y+110+2z)} = \frac{v-u}{36z^2}; \quad (3)$$

Again, time of filling in first irruption was

$$\frac{2z^2(x+10z)}{u} = \frac{2}{3}\left\{\frac{2z^2(y+10z)}{u}\right\} \text{ by question;} \quad (4)$$

and, by the last *datum*,

$$\frac{2z^2(x+10z)}{u} = \frac{2z^2(y+10z)}{v} - 10. \quad (5)$$

By means of (1), eliminate $u$ and $v$ from (2), by writing for them the proportional quantities $16z$ and $y+2z$; then

$$\frac{8}{x+2z} - \frac{1}{2z} = \frac{y-14z}{36z^2}, \quad \text{or} \quad \frac{14z-x}{x+2z} = \frac{y-14z}{18z}; \quad (a)$$

similarly, (3) gives

$$\frac{1}{2z} - \frac{y+2z}{2z(y+110+2z)} = \frac{y-14z}{36z^2}, \quad \text{or} \quad \frac{110}{y+110+2z} = \frac{y-14z}{18z}; \quad (\beta)$$

from (4) we get $x + 10z = \frac{2}{3}(y+10z)$; $(\gamma)$

$\therefore \frac{x}{z} = \frac{2y-10z}{3z}$, whence by [27] we get in $(a)$

$$\frac{42z-(2y-10z)}{(2y-10z)+6z} \text{ or } \frac{26z-y}{y-2z} = \frac{y-14z}{18z}, \text{ whence } y = 20z;$$

$\therefore (\beta)$ gives $z = 10$, $y = 200$, and $(\gamma)$ $x = 100$;

hence (1) gives $u = \frac{8}{11}v$, and (5) $v = 500$;

$\therefore$ the second time of filling $= \frac{2z^2(y+10z)}{v} = 120' = 2$ hrs.

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12. 3. By (85) $\dfrac{a^{\frac{r}{s}}+1}{a^{\frac{r}{s}}-1} = \dfrac{x^{(m-n)^2}}{x^{-4mn}} = x^{(m-n)^2+4mn} = x^{(m+n)^2}$.

4. From (2) $xy + 10\sqrt{(xy)} + 25 = 16(y+2)$,

or $\sqrt{(xy)} + 5 = 4\sqrt{(y+2)} = 4z$ suppose:

whence $\frac{9}{64}x^2 - x + \frac{16}{9} = 9z^2 - 26z + \frac{169}{9}$, or $\frac{3}{8}x - \frac{4}{3} = 3z - \frac{13}{3}$;

$\therefore z = \frac{1}{8}x + 1 = \sqrt{(y+2)}$, and $y = \frac{1}{64}x^2 + \frac{1}{4}x - 1$;

also $\sqrt{(xy)} = 4z - 5 = \frac{1}{2}x - 9$, or $(\frac{1}{2}x - 9)^2 = xy = x(\frac{1}{64}x^2 + \frac{1}{4}x - 1)$.

5. Let $x$ = first majority; $\therefore$ $600 - x$ = minority, and the question was lost by $x - (600 - x) = 2x - 600$: afterwards it was carried by $4x - 1200$;

$$\therefore \text{ new minority} = \tfrac{1}{2}\{600 - (4x - 1200)\} = 900 - 2x,$$

$$\text{and new majority} = 600 - (900 - 2x) = 2x - 300:$$

hence $2x - 300 : x :: 8 : 7$, whence $x = 350$, who voted against it the first time, and $900 - 2x = 200$, who voted against it the second time.

6. $AB = x$, $BC = y$, $z$ = rate of pedestrian, $3z$ of coach: then

$$\frac{x+y}{z} = \frac{\sqrt{(x^2+y^2)}}{z} + 2\tfrac{2}{3}; \quad (1)$$

now he has gone $x + 8$ miles when the coach overtakes him, and for this he took $\frac{x+8}{z}$ hours; $\therefore$ the coach took $\frac{x+8}{z} - 4$ hours, and $\therefore$ $3z\left(\frac{x+8}{z} - 4\right) = x + 8$: (2)

also the coach, starting 4 hours after him, does the whole circuit twice over, besides resting $6\frac{2}{3}$ hours, while the man does it once and rests 4 hours;

$$\therefore 4 + \frac{2\{x+y+\sqrt{(x^2+y^2)}\}}{3z} + 6\tfrac{2}{3} = \frac{x+y+\sqrt{(x^2+y^2)}}{z} + 4. \quad (3)$$

From (3) $x+y+\sqrt{(x^2+y^2)} = 20z$, from (1) $x+y-\sqrt{(x^2+y^2)} = \frac{8}{3}z$;

$$\therefore x + y = \tfrac{34}{3}z, \quad \sqrt{(x^2+y^2)} = \tfrac{26}{3}z,$$

$$\text{and} \quad \frac{x+y}{\sqrt{(x^2+y^2)}} = \frac{17}{13}, \quad \text{whence } \frac{y}{x} = \frac{12}{5}.$$

(We take this value, and not $\frac{5}{12}$, because $AB$ is given $< BC$,)

$$\therefore \frac{x+y}{x} = \frac{17}{5} = \frac{34z}{3x}, \qquad \text{and } x = \tfrac{10}{3}z:$$

substitute this value for $x$ in (2).

7. Let $x$, $y$, $z$ be the weights of $A$, $B$, $D$ in carats when rough; $\therefore$ $\frac{1}{2}x$, $\frac{1}{2}y$, are the weights of dust from $A$ and $B$: hence $x - 52$ = value of $C$ in £'s, and therefore

$$C\text{'s weight} : 1 :: \sqrt{(x - 52)} : \sqrt{2};$$

hence $C$'s weight $= \sqrt{(\frac{1}{2}x - 26)}$, and $(x - 52) + 2z^2 = y$. (1

Also value of dust from $A$ and $B = \frac{1}{2}(x+y) = 85$; (2)
and $3x^2 : 3(\frac{1}{2}x - 26) + 3z^2 + \frac{1}{2}x :: \frac{3}{2}y^2 : \frac{1}{2}y :: 3y : 1$. (3)

From (1) $y - x = 2z^2 - 52$, from (2) $y + x = 170$;

$$\therefore\ x = 111 - z^2,\quad y = 59 + z^2:$$

simplify (3), and substitute for $x$ and $y$ in it.

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**13.** 3. $x^2 - 2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + 16 = 7x$, or $(x^2 - 2x^{\frac{3}{2}} + x) - 8(x - x^{\frac{1}{2}}) + 16 = 16x^{\frac{1}{2}}$;

$$\therefore\ (x - x^{\frac{1}{2}}) - 4 = 4x^{\frac{1}{4}},\ \text{ or } 4(x^{\frac{1}{4}} + 1) = x^{\frac{1}{2}}(x^{\frac{1}{2}} - 1);$$

$$\therefore\ x^{\frac{1}{4}} + 1 = 0 \text{ and } x = 1;\ \text{ or } 4 = x^{\frac{1}{2}}(x^{\frac{1}{4}} - 1),$$

$$\text{that is, } (x^{\frac{1}{4}} - 2)(x^{\frac{1}{2}} + x^{\frac{1}{4}} + 2) = 0, \text{ \&c.}$$

4. From (1) $x^2y + xy^2 + \frac{1}{4}y^3 = xy^2 + 4x^{\frac{1}{2}}y + 4$, or $xy^{\frac{1}{2}} + \frac{1}{2}y^{\frac{3}{2}} = x^{\frac{1}{2}}y + 2$,

$$\text{whence } x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}}) = 2 - \tfrac{1}{2}y^{\frac{3}{2}} = x^{\frac{3}{2}} - 3 \text{ by (2):}$$

$$\text{hence } x^{\frac{3}{2}} - y^{\frac{3}{2}} = 5 - \tfrac{3}{2}y^{\frac{3}{2}},\ \text{ and } x^{\frac{3}{2}} - y^{\frac{3}{2}} - 3x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}})$$

$$= (5 - \tfrac{3}{2}y^{\frac{3}{2}}) - 3(2 - \tfrac{1}{2}y^{\frac{3}{2}}),\ \text{ or } (x^{\frac{1}{2}} - y^{\frac{1}{2}})^3 = -1;$$

$\therefore\ x^{\frac{1}{2}} - y^{\frac{1}{2}} = -1$, whence (2) becomes $x^{\frac{3}{2}} - 3 = -x^{\frac{1}{2}}y^{\frac{1}{2}} = -x - x^{\frac{1}{2}}$

or $x^{\frac{3}{2}} + x + x^{\frac{1}{2}} = 3$, which has manifestly one solution, $x^{\frac{1}{2}} = 1$

$$\therefore\ y^{\frac{1}{2}} = x^{\frac{1}{2}} + 1 = 2.$$

5. Let $x$ = length of course in yards; then $x - 2$ = n° of yards A has gone (uniformly) in 6′, and $\therefore\ \frac{1}{6}(x-2)$ = A's rate per minute: hence in 4′ A will have gone $\frac{2}{3}(x-2)$ and B $\frac{2}{3}(x-2) - \frac{1}{440}x$, and $\therefore$ B's first rate $= \frac{1}{4}\{\frac{2}{3}(x-2) - \frac{1}{440}x\}$, at which rate B goes for 5′, but in the last minute makes this and 20 yards more:

$$\therefore\ \tfrac{6}{4}\{\tfrac{2}{3}(x-2) - \tfrac{1}{440}x\} + 20 = \text{whole course} = x.$$

Otherwise: let $x$, $y$ be A's and B's rates per minute at first; then length of course $= 6x + 2$:

and $4(x-y) = \frac{1}{440}(6x+2)$, (1) and $5y + (y+20) = (6x+2)$. (2)

6. Let $x$ = revenue before increase, $m\sqrt{x}$ = expense of collecting, $y$ = interest of debt: then $\frac{9}{4}x$ = increased revenue; and net income $= \frac{9}{4}x - (\frac{3}{2}m\sqrt{x} + y) = \frac{81}{23}\{x - (m\sqrt{x} + y)\}$: (1) also $\frac{9}{16}x$ = reduced income, and

$$\tfrac{9}{16}x - (\tfrac{3}{4}m\sqrt{x} + y) = \tfrac{8}{23}\{x - (m\sqrt{x} + y)\} = 4. \quad \text{(2) and (3)}$$

From (1) and (3), eliminating $m\sqrt{x}$, we get $3x + 2y = 248$, from (2) and (3), $3x + 4y = 304$, whence $2y = 56$, and $x = 64$ (millions).

7. $x$ = A's or C's n° of strokes per minute at starting, $y$ = n° of yards in each of C's, $y + 1$ of A's: $\therefore \frac{7}{6}x$ = n° of strokes made by B, and if $z$ = n° of yards in each stroke, then their velocities at starting are as $x(y+1)$, $\frac{7}{6}xz$, $xy$, or as $6(y+1)$, $7z$, $6y$, respectively.

Now when A has made 42 strokes, B has made 49, whose lengths form the A. P. $z$, $z + \frac{1}{12}$, &c, whose sum $= 49(z+2)$, the last term being $z + 4$; and now he has gained 19 yards upon $A$, (who will have made $42(y+1)$ yards) besides having lost 16 yards by steering:

hence $49(z+2) = 42(y+1) + 19 + 16$, or $7z + 3 = 6y$. (1)

In this time C has made 42 yards less than A, and $\therefore$ will have lost $42 + 19 = 61$ yards upon B, and, having been at first 20 yards behind him, will now be 81 yards behind: but C now begins to make as many strokes per minute as B; and in 28 strokes B makes $(z+4) + (z + 4 - \frac{1}{6}) +$ &c. $= 7(4z+7)$ yards, of which it loses 11 by steering; and C makes $y + (y + \frac{1}{6}) +$ &c. $= 7(4y+9)$ yards:

$\therefore 7(4y+9) = 7(4z+7) - 11 + 81$, whence $y = z + 2$. (2)

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**14.** 2. The product gives

$$a^4 - \tfrac{3}{16}b^4 = (a^2 + \tfrac{1}{2}bx - \tfrac{1}{4}b^2)^2 - [a\sqrt{\{b(x+b)\}} - \tfrac{1}{2}b\sqrt{(x^2 - bx + b^2)}]^2$$
$$= a^4 + a^2bx - \tfrac{1}{2}a^2b^2 + \tfrac{1}{4}b^2x^2 - \tfrac{1}{4}b^3x + \tfrac{1}{16}b^4 - a^2b(x+b)$$
$$+ ab\sqrt{\{b(x^3 + b^3)\}} - \tfrac{1}{4}b^2(x^2 - bx + b^2),$$

which reduces to $\frac{3}{2}ab = \sqrt{\{b(x^3 + b^3)\}}$.

3. Squaring we get $a^2(x^8 + 1) + 4(x^6 - x^2) + 2a^2x^4 = 0$, an ordinary recurring equation;

whence $a^2z^2 + 4z + 4a^2 = 0$, if $z = x^2 - x^{-2}$.

4. From (1) $3x^4 - 12x^3y + 10x^2y^2 - 6x^2 + 8xy + 1 = 0$,

or $3\{(x^4 - 4x^3y + 4x^2y^2) - 2(x^2 - 2xy) + 1\} = 2\{x^2y^2 + 2xy + 1\}$,

whence $\sqrt{3}(x^2 - 2xy - 1) = \pm\sqrt{2}(xy + 1)$; (3)

also from (2) $x^4 + 1 = 2y^4$: (4)

therefore (1) gives

$$(2 + 4xy - 3x^2)^2 = 2 - 4x^2y^2 + x^4 + 2x^4 = (x^2 - 2y^2)^2 \text{ by } (4),$$

and $2 + 4xy - 3x^2 = \pm (x^2 - 2y^2)$, whence (with lower sign)

$$x - y = \pm 1.$$

Substituting for $y$ in 3, we get (when simplified)

$$(\sqrt{3} \pm \sqrt{2})\, x^2 \mp (2\sqrt{3} \pm \sqrt{2})\, x + (\sqrt{3} \pm \sqrt{2}) = 0,$$

$$\text{or } \; x^2 \mp (4 - \sqrt{6})\, x + 1 = 0.$$

5. Let £$x$ = cost of inflation, $y$ = n° of spectators the enclosure would hold, £$z$ = charge for ascending:

$$\text{then } \; \tfrac{1}{10}y + 2z = \tfrac{12}{5}(x + 5); \quad (1)$$

$$\tfrac{1}{40}y = \tfrac{2}{3}(\tfrac{1}{2}x + 5) = \tfrac{1}{3}(x + 10), \quad (2)$$

and he has now lost $\tfrac{1}{3}(\tfrac{1}{2}x + 5) = \tfrac{1}{6}(x + 10)$;

$$\therefore \; \tfrac{3}{40}y + z = (x + 5) + \tfrac{1}{6}(x + 10) + 10.$$

6. Let the distance = 1, rate of stream = $a$, A's rate = $x$, B's = $y$; time by A's watch = $rt$, by B's = $st$, where $t$ is the true time: then time down by B $= \dfrac{1}{y + a}$, and apparent time to him $= \dfrac{r}{y + a}$; also when A's app. time is $\dfrac{r}{y + a}$, the real time is $\dfrac{r}{s(y + a)}$, and space described by him $= \dfrac{r(x + a)}{s(y + a)}$.

$$\text{hence } \; \frac{r}{s(y + a)} = 6\left\{1 - \frac{r(x + a)}{s(y + a)}\right\} \div \frac{2}{3}(x + a) = \frac{9}{x + a} - \frac{9r}{s(y + a)},$$

$$\text{or } \; \frac{x + a}{y + a} = \frac{9s}{10r}. \quad (1)$$

$$\text{Similarly, } \; \frac{r}{s(y - a)} = \left\{1 - \frac{r(x - a)}{s(y - a)}\right\} \div \frac{2}{3}(x - a) = \frac{3}{2(x - a)} - \frac{3r}{2s(y - a)},$$

$$\text{or } \; \frac{x - a}{y - a} = \frac{3s}{5r}. \quad (2)$$

$$\text{Also } \; \frac{s}{r(y - a)} = \frac{5}{7(x - a)} - \frac{5s}{7r(y - a)}, \quad \text{or } \; \frac{x - a}{y - a} = \frac{5r}{12s}. \quad (3)$$

From (2) and (3) $\dfrac{r}{s} = \dfrac{6}{5}$, and then $x = 2a$, $y = 3a$.

7. On leaving barracks, let $x$ be the n° of Officers, $2x$ of Serjeants, $y$ of Privates;

$$\text{then } 100 \times (2x) + x = 10y + 20. \quad (1)$$

Now, in street-fighting, there fell 2 O,

and $\therefore\ 2 \times \frac{2x}{x} \times (2)$ or 8 S, and $2 \times \frac{y}{x} \times (2)$ or $\frac{4y}{x}$ P;

and there remain $(x-2)$ O, $2(x-4)$ S, $\frac{y(x-4)}{x}$ P:

at the barricades there fell 3 O,

and $\therefore\ \frac{3}{4} \times \frac{2(x-4)}{x-2} \times (3)$ or $\frac{9(x-4)}{2(x-2)}$ S, and $\frac{9y(x-4)}{4x(x-2)}$ P,

and there are now (including the volunteer officer)

$$(x-4)\text{ O},\quad \frac{(x-4)(4x-17)}{2(x-2)}\text{ S},\quad \frac{y(x-4)(4x-17)}{4x(x-2)}\text{ P}:$$

in the retreat there fell 3 O,

and $\therefore\ \frac{4}{3} \times \frac{(x-4)(4x-17)}{2(x-2)(x-4)} \times (3)$ or $\frac{2(4x-17)}{x-2}$ S, $\frac{y(4x-17)}{x(x-2)}$ P,

and $\therefore$ there return to barracks (omitting the volunteer officer),

$$(x-8)\text{ O},\quad \frac{(4x-17)(x-8)}{2(x-2)}\text{ S},\quad \frac{y(4x-17)(x-8)}{4x(x-2)}\text{ P}:$$

hence $100 \times \frac{(4x-17)(x-8)}{2(x-2)} + (x-8) = 10 \times \frac{y(4x-17)(x-8)}{4x(x-2)} + 12.$ (2)

From (2) $12 = (x-8)\{1 + \frac{4x-17}{4x(x-2)}(200x - 10y)\}$

$$= \text{by (1) } (x-8)\{1 + \frac{(4x-17)(20-x)}{4x(x-2)}\} = (x-8)\frac{89x-340}{4x(x-2)},$$

$$\text{or } 41x^2 - 956x + 2720 = 0, \quad \text{whence } x = 20.$$

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**15** 3. $(x^2 - ax + a^2) + \sqrt{\{x(x+a)(x^2 - ax + a^2)\}} + \frac{1}{4}x(x+a) = \frac{9}{4}x(x+a);$

$$\therefore \sqrt{(x^2 - ax + a^2)} = (-\tfrac{1}{2} \pm \tfrac{3}{2})\sqrt{(x^2 + ax)} = \sqrt{(x^2 + ax)}$$
$$\text{or } -2\sqrt{(x^2 + ax)}.$$

4 In (2) put $x = v^2 y$; then $3v^4 - 20v^3 + 42v^2 - 44v + 16 = 0$,

$$\begin{aligned} \text{or } 3(v-1)^4 - 8(v-1)^3 - 8(v-1) - 3 &= 0 \\ &= 3\{(v-1)^4 - 1\} - 8(v-1)\{(v-1)^2 + 1\} \\ &= \{(v-1)^2 + 1\}[3\{(v-1)^2 - 1\} - 8(v-1)]: \end{aligned}$$

the factor $(v-1)^2+1=0$ gives $v$ impossible; but the other factor gives $3v^2-14v+8=0$, whence $v=4$ or $\frac{2}{3}$: taking $v=\frac{2}{3}$ in (1), we get

$$\frac{27y^{\frac{3}{2}}-1}{\frac{64}{729}y^3+3y^2-\frac{8}{9}y^{\frac{5}{2}}}=27\left(\frac{x}{y}\right)^{\frac{3}{2}}=8,$$

$$\text{or}\ \ \frac{512}{729}y^3-\frac{64}{9}y^{\frac{5}{2}}+24y^2-27y^{\frac{3}{2}}=-1,$$

where extract the cube root.

5. Let $x$ = rent in £'s, $y$ = n° of acres:

$$\text{then}\ \ \frac{1}{10}x=\frac{3}{20}y+6,\ \ (1)\qquad \frac{3}{20}x=\frac{6}{20}y.\ \ (2)$$

6. Let $x$ = distance from C to L, $y$ = rate of T, $z$ of F: then in $2\frac{1}{2}$ hrs A goes $2\frac{1}{2}\times 3\frac{1}{2}=\frac{35}{4}$ miles, and T does the same distance in $\dfrac{35}{4y}$ hrs; ∴ the time is now $(6+\dfrac{35}{4y})$ A.M., and A left C $2\frac{1}{2}$ hrs before, that is at $(3\frac{1}{2}+\dfrac{35}{4y})$ A.M., and consequently, by the time F passes, at $9\frac{5}{6}$ A.M., has been $(6\frac{1}{3}-\dfrac{35}{4y})$ hrs on the road, and F has been $9\frac{5}{6}-7\frac{1}{2}=2\frac{1}{3}$ hrs;

$$\text{hence}\ \ 2\tfrac{1}{3}z=\left(6\tfrac{1}{3}-\frac{35}{4y}\right)3\tfrac{1}{2}.\quad (1)$$

On the return, F has been $2\frac{1}{3}$ hrs on the road when it meets him at a point, therefore, which is $2\frac{1}{3}z$ miles from L: hence A, since starting, has travelled $x-2\frac{1}{3}z$ miles, besides resting $2\frac{1}{2}$ hrs, and has therefore consumed

$$\{(x-2\tfrac{1}{3}z)\div 3\tfrac{1}{2}+2\tfrac{1}{2}\}\ \text{hrs}=(\tfrac{2}{7}x-\tfrac{2}{3}z+2\tfrac{1}{2})\ \text{hrs}:$$

the time after noon is therefore

$$\left(3\tfrac{1}{2}+\frac{35}{4y}\right)+(\tfrac{2}{7}x-\tfrac{2}{3}z+2\tfrac{1}{2})-12=5\tfrac{1}{3},\ \text{by question.}\quad (2)$$

Also half-a-mile further back he met T, which was therefore $2\frac{1}{3}z+\frac{1}{2}$ miles from L, and since A had taken $\frac{1}{2}\div 3\frac{1}{2}=\frac{1}{7}$ hr for this, the time was $5\frac{1}{3}-\frac{1}{7}=5\frac{4}{21}$, and T had been $2\frac{4}{21}$ hrs on the road;

$$\therefore\ 2\tfrac{4}{21}y=2\tfrac{1}{3}z+\tfrac{1}{2},\ (3)=\left(6\tfrac{1}{3}-\frac{35}{4y}\right)3\tfrac{1}{2}+\tfrac{1}{2},\ \text{by (1)},$$

whence $y=8\frac{3}{4}$, and then from (1) $z=8$, and from (2) $x=54\frac{5}{6}$.

7. Let $P$, $Q$, $R$, be the plumpers for A, B, C; $p$, $q$, $r$, the votes split between B and C, A and C, A and B: then the poll stood at first, for A, $P+q+r$, for B, $Q+p+r$, for C, $R+p+q$.

$$\text{Now}\quad 3R = P+Q+R+p+q+r, \quad (1)$$

$$P+Q = \tfrac{4}{21}(P+Q+R+2p+2q+2r), \ (2) \quad p = 2q; \ (3)$$

$$(Q+p+r)-(R+p+q) = 110. \quad (4)$$

After the scrutiny, the poll stands

$$\text{for A, } P+q, \quad \text{for B, } Q+p, \quad \text{for C, } R+p+q:$$

$$\text{and}\quad R+p+q = P+q+200, \ (5) \quad P = q. \ (6)$$

From (5), with (3) and (6), $R = 200 - q$, whence (4) gives $Q = 310 - r$, putting which values in (4) we get $q = 15$;

$$\therefore P = 15, \quad R = 185, \quad p = 30;$$

and since (2) gives $17(P+Q) = 4(R+2p+2q+2r)$, putting $Q = 310 - r$, we find $r = 177$, $Q = 133$.

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**16.** 3. Since $\sqrt[4]{(x^3-3x+2)} = \sqrt[4]{\{(x^2-2x+1)(x+2)\}} = \sqrt{(x-1)}\,.\,\sqrt[4]{(x+2)}$, we get $(x-1) - \sqrt{(x-1)}\,.\,\sqrt[4]{(x+2)} + \frac{1}{4}(x+2) = \frac{9}{4}(x+2)$,

$$\text{whence}\quad \sqrt{(x-1)} = 2\sqrt[4]{(x+2)} \quad\text{or}\quad -\sqrt[4]{(x+2)}.$$

4. From the form of the equation, assume

$$x^2 = \frac{m^2-1}{m^2+1}, \quad y^2 = \frac{n^2-1}{n^2+1}:$$

then (2) gives $4\sqrt{\dfrac{(m^2-1)(n^2-1)}{(m^2+1)(n^2+1)}} = \dfrac{4\sqrt{2}}{(m^2+1)(n^2+1)}$,

$$\text{whence}\quad (m^4-1)(n^4-1) = 2, \quad\text{and}\quad m^2 = \sqrt{\frac{n^4+1}{n^4-1}}.$$

Also (1) gives

$$\frac{8n^2}{(m^2+1)^2(n^2+1)} - \frac{8m^4}{(m^2+1)^2(n^2+1)} = 4\,\frac{(m^2-1)\sqrt{\{2(n^4+1)\}}}{(m^2+1)(n^2+1)},$$

$$\text{whence}\quad 2n^2 - 2m^4 = (m^4-1)\sqrt{\{2(n^4+1)\}};$$

substitute for $m^2$ in this, and square;

$$\therefore n^{12} - 2n^{10} - n^8 + n^4 + 2n^2 - 1 = 0,$$

a recurring equation in $n^2$, which [61] is div. by $n^4-1$: effecting this division we get

$$n^8 - 2n^6 - 2n + 1 = 0, \text{ whence } (n^4+2+n^{-4}) - 2(n^2+n^{-2}) + 1 = 3;$$

$$\therefore n^2 + n^{-2} = 1 \pm \sqrt{3}, \ \&c.$$

5. Let $x$ = n° of hrs before noon at which A began, $x-\frac{1}{2}$ = do. for B: then A took $(x+9\frac{3}{4})-1=\frac{1}{4}(4x+35)$ hrs to finish his half, and B $\{(x-\frac{1}{2})+7\}-1=\frac{1}{2}(2x+11)$ hrs to finish his: ∴ in *one* hour A does $\frac{4}{4x+35}$ of half the whole, or $\frac{2}{4x+35}$ of the whole, and B $\frac{1}{2x+11}$ of it:

$$\text{hence } \frac{2x}{4x+35}+\frac{(x-\frac{1}{2})}{2x+11}=\frac{1}{2}.$$

6. Let A with vel. $13v$ run from $C$ to $A$, B, with vel. $11v$ from $C$ to $B$: let them meet first at $D$, the middle of $AB$, and again at $E$ in $CA$, where $CE=30$; $CA=x$, $CB=y$:

$$\text{then } \frac{CA+AD}{13v}=\frac{CB+BD}{11v}, \text{ or } \frac{x+\frac{1}{2}\sqrt{(x^2+y^2)}}{y+\frac{1}{2}\sqrt{(x^2+y^2)}}=\frac{13}{11}; \quad (1)$$

$$\text{and } \frac{CA+AB+BC+CE}{13v}=\frac{CB+BA+AE}{11v},$$

$$\text{or } \frac{x+y+\sqrt{(x^2+y^2)}+30}{x+y+\sqrt{(x^2+y^2)}-30}=\frac{13}{11}. \quad (2)$$

From (2) $x+y+\sqrt{(x^2+y^2)}=360$, from (1) $\frac{x+y+\sqrt{(x^2+y^2)}}{x-y}=\frac{24}{2}$,

$$\text{or } \frac{360}{x-y}=12, \text{ whence } x-y=30:$$

substitute for $y$ in $\sqrt{(x^2+y^2)}=360-x-y$, and square, &c.

7. Let $x$, $50-x$ be the prices given in £'s, $2x$, $25-\frac{1}{2}x$ those received; $y$ the per centage on price which A cost in keep, and B produced in hire; then A produced in hire $\frac{1}{2}x$, and cost in keep $\frac{1}{100}xy$; B produced in hire $\frac{1}{100}(50-x)y$, and cost in keep $\frac{5}{8}\times(25-\frac{1}{2}x)$ guineas $=\frac{5}{8}\times\frac{21}{20}(25-\frac{1}{2}x)$ £'s:

$$\therefore \frac{1}{100}xy+\frac{5}{8}\times\frac{21}{20}(25-\tfrac{1}{2}x)=33; \quad (1)$$

and profit upon (produce – keep) =, by question, *eight* times profit on sale;

$$\therefore \{\tfrac{1}{2}x+\tfrac{1}{100}(50-x)y\}-33=8[\{2x+(25-\tfrac{1}{2}x)\}-\{x+(50-x)\}]. \quad (2)$$

$$\text{From (1) } y=\frac{75}{16}\times\frac{7x+354}{x}=50\times\frac{23x-334}{50-x} \text{ from (2)},$$

$$\text{whence } 757x^2-10676x-53100=0, \text{ and } x=18.$$

**17.** 2. Divide (1) by $a^n b^m$;

$$\text{then } \left(\frac{x}{b}\right)^m + \left(\frac{y}{a}\right)^n = 2\sqrt{\frac{a^m x^m b^n y^n}{a^{2n} b^{2m}}} = 2\sqrt{\left\{\left(\frac{a}{b}\right)^{m-n}\left(\frac{x}{b}\right)^m\left(\frac{y}{a}\right)^n\right\}};$$

$$\text{but (2) } \frac{x}{b} = \frac{a}{y} = z, \text{ suppose;}$$

$$\therefore z^m + z^{-n} = 2\left(\frac{a}{b}\right)^{\frac{m-n}{2}} z^{\frac{m-n}{2}}, \text{ or } z^{\frac{m+n}{2}} + z^{-\frac{m+n}{2}} = 2\left(\frac{a}{b}\right)^{\frac{m-n}{2}},$$

$$\text{or } z^{m+n} - 2\left(\frac{a}{b}\right)^{\frac{m-n}{2}} z^{\frac{m+n}{2}} + 1 = 0, \text{ \&c:}$$

and, $x$ being found, $y$ may be obtained from (2).

3. $(x^2 + x + 1) - 2\sqrt{\{(x-1)(x^2+x+1)\}} + x - 1 = 4(x-1)$;

$$\therefore \sqrt{(x^2 + x + 1)} = \sqrt{(x-1)}\,(1 \pm 2) = \text{\&c.}$$

4. Put $y = xz$; then (2) gives $x^2 z^2 = a^2(z^4 + 2z^2 - 1)$;

and (1) $a^2(1+z^4) = x^2 z^6 - 2a^2 z\sqrt{(1-z^4)} = a^2 z^4\{z^4 + 2z^2 - 1\} - 2a^2 z\sqrt{(1-z^4)}$,

or $z^8 - 1 + 2z^4(z^2-1) - 2z\sqrt{(1-z^4)} = 0$; $\therefore z = 1$, $x = y = \pm a\sqrt{2}$.

5. Let $x$ = length of course, $y$ = A's pace, $z$ = B's; then A runs to the post and 90 yards back again while B has still 90 yards to go to the post, that is, A has gone $x + 90$ yards while B has gone $x - 90$;

$$\therefore \frac{x+90}{x-90} = \frac{y}{z}: \quad (1)$$

also A takes 3′ less than B in running the distance $2x$:

$$\therefore \frac{2x}{y} = \frac{2x}{z} - 3: \quad (2)$$

and A would have gone $2x + \frac{1}{6}x = \frac{13}{6}x$ yards while B would have gone $2x - \frac{1}{6}x = \frac{11}{6}x$ yards;

$$\therefore \frac{13}{11} = \frac{y}{z} = \frac{x+90}{x-90}, \text{ whence } x = 1080.$$

6. Let $x$ = n° of spokes in fore-wheel, $y$ in hind-wheel: then $\frac{2\pi R}{x}$ = arc between spokes of former, $\frac{2\pi r}{y}$ = do. of latter;

$$\therefore 2\pi r = \tfrac{3}{4} \cdot 2\pi R - \frac{2\pi R}{y}, \text{ or } \frac{1}{y} = \frac{3}{4} - \frac{r}{R}: \quad (1)$$

$$\text{also } \tfrac{9}{8} \cdot 2\pi R = \tfrac{7}{4} \cdot 2\pi r - \tfrac{1}{2} \cdot \frac{2\pi r}{x}, \text{ or } \frac{1}{x} = \frac{7}{2} - \frac{9}{4} \cdot \frac{R}{r}; \quad (2)$$

$$\text{and } 2r : R - r :: x : 2. \quad (3)$$

7. Let $x$ = rental, $y$ the per centage allowed on rents paid, $y$ the rents paid in second year; then he embezzles £270, besides obtaining the per centage on the amount, $y-270$, which he returns;

$$\therefore 270 + r\,.\,\frac{y-270}{100} - \frac{1}{2}r\,.\,\frac{x-(y-270)}{100} = 1^{\text{st}}\text{ year's income} = \frac{6x}{100};\ (1)$$

in $3^{\text{rd}}$ year the rental is $\frac{92\frac{1}{2}}{100}x = \frac{37}{40}x$, the amount not paid is $x-y$, and the amount paid $\frac{37}{40}x - (x-y) = y - \frac{3}{40}x$;

$$\therefore r\,.\,\frac{y-\frac{3}{40}x}{100} - \frac{1}{2}r\,.\,\frac{x-y}{100} = \frac{3}{4}\,.\,\frac{6x}{100},\ (2)\quad\text{and def}^{y} = \frac{1}{4}\,.\,\frac{6x}{100};$$

$$\therefore \frac{1}{4}\,.\,\frac{6x}{100} = 2\{270 - r\frac{270}{100} - \frac{r}{2}\,.\,\frac{270}{100}\},\ \text{or}\ x = 180\,(200-3r).\quad (3)$$

From (1) we have $(270 - r\,.\,\frac{270}{100} - \frac{r}{1}\,.\,\frac{270}{100}) + \frac{1}{2}r\,.\,\frac{3y-x}{100} = \frac{6x}{100}$,

$$\text{or by (3)}\ \frac{3x}{400} + \frac{1}{2}r\,.\,\frac{3y-x}{100} = \frac{6x}{100},$$

$$\text{whence}\ \frac{3y}{x} = 1 + \frac{21}{2r} = \frac{23}{20} + \frac{9}{r}\ \text{from (2).}$$

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**18.** 3. Dividing by $1+x$, (and $\therefore x = -1$ is a root) we have

$$1 - x + x^2 - x^3 + x^4 = a\,(1+x)^4 = a\,(1 + 4x + 6x^2 + 4x^3 + x^4);$$

$$\therefore (a-1)\,(x^4+1) + (4a+1)\,(x^3+x) + (6a-1)\,x^2 = 0,$$

or $(a-1)\,(x^4+2x^2+1) + (4a+1)\,x\,(x^2+1) + (4a+1)\,x^2 = 0$,

or $(a-1)\,z^2 + (4a+1)\,z + (4a+1) = 0$, if $z = x + x^{-1}$; &c.

4. From (1)

$$\{\sqrt[3]{y} - \sqrt[3]{(a+bx)}\}^3 = a - bx,\ \text{or}\ \sqrt[3]{y} = \sqrt[3]{(a+bx)} + \sqrt[3]{(a-bx)}:$$

$$\therefore y = 2a + 3\sqrt[3]{(a^2-b^2x^2)}\,\{\sqrt[3]{(a+bx)} + \sqrt[3]{(a-bx)}\}$$

$$= 2a + 3\sqrt[3]{y}\,\sqrt[3]{(a^2-b^2x^2)};$$

whence (2) gives $\frac{2a}{\sqrt[3]{(a-bx)}} = 2a\sqrt[3]{(a-bx)}$, or $x = \frac{a-1}{b}$.

5. Let $x$ = time in hours for B, $y$ for $S$: then B drinks $\frac{1}{x}$ of wine in cask in one hour, and $\therefore \frac{2y}{3x}$ in $\frac{2}{3}y$ hours,

leaving $1 - \frac{2}{3}\frac{y}{x}$ of it, which *S* drinks in $(1 - \frac{2y}{3x}) \div \frac{1}{y}$ hrs: hence $\frac{2}{3}y + (1 - \frac{2y}{3x})\,y$ = time in which cask was emptied, which, by question, exceeds by 2 hours the time in which they would together have emptied it; or

$$\tfrac{2}{3}y + (1 - \frac{2y}{3x})\,y = 1 \div (\frac{1}{x} + \frac{1}{y}) + 2 = \frac{xy}{x+y} + 2; \quad (1)$$

and $\frac{xy}{x+y} \times \frac{1}{x}$ = what B would have drunk $= \frac{1}{2}(1 - \frac{2y}{3x})$. (2)

From (2) $3x^2 - 5xy - 2y^2 = 0$, whence $x = 2y$:

from (1) $y\{\frac{2}{3} + 1 - \frac{2}{3}\frac{y}{x} - \frac{x}{x+y}\} = y\{\frac{2}{3} + 1 - \frac{1}{3} - \frac{2}{3}\} = 2$,

or $y = 3$, $x = 6$.

6. Let $x$ = n° of acres, $y$ = (in shillings) agreed price per acre, $r$ = reduction per acre, $\frac{1}{2}z$, $\frac{1}{3}z$, $\frac{1}{4}z$, n° of acres that can be reaped per day by A, B, C. Then in two days A, B, C will have reaped $z + \frac{2}{3}z + \frac{1}{2}z = \frac{13}{6}z$, and C will receive

$$\tfrac{1}{2}z\,(y - r) = 5. \quad (1)$$

Of the remaining $x - \frac{13}{6}z$, A and B do in the ratio of $\frac{1}{2}$ and $\frac{1}{3}$, that is, A does $\frac{3}{5}$, B $\frac{2}{5}$; hence

$$\tfrac{3}{5}(x - \tfrac{13}{6}z) = \text{A's work for one day} = \tfrac{1}{2}z, \quad (2)$$

and A receives altogether $z\,(y - r) + \frac{1}{2}zy$. But, if C had remained, A would have done $\frac{6}{13}$ of the whole work, and would have received $\frac{6}{13}xy$;

$$\therefore z\,(y - r) + \tfrac{1}{2}zy = \tfrac{6}{13}xy - 1\tfrac{1}{2}; \quad (3)$$

and the farmer saves $r$ shillings on each acre of the first two days' work, that is, he saves (in shillings)

$$\tfrac{13}{6}zr = \text{(by question) } \tfrac{1}{2}y. \quad (4)$$

From (2) $x = 3z$; putting this in (3), we get $26rz - 3yz = 39$; combining this with (1), $yz - rz = 10$, we get $rz = 3$, $yz = 13$; hence by (4) $y = \frac{13}{3}rz = 13$; $\therefore z = 1$, $r = 3$, $x = 3$.

7. Let $P, Q, R$, denote the plumpers for P, Q, R; $p, q, r$, the split votes between Q and R, P and R, P and Q, respectively: then the whole number of votes was $P + (q + r)$ for P, $Q + (p + r)$ for Q, and $R + (p + q)$ for R: hence we have the equations

$$P + Q + R = 565, \ (1) \quad p + q + r = 776, \ (2) \quad Q + 1 = q - p, \ (3)$$

$$p = 2r + 1, \ (4) \quad Q + p + r = P + q + r - 5 = R + p + q - 48. \ (5) \text{ and } (6)$$

From (5) $P-5=Q+p-q=-1$ by (3); $\therefore\ P=4$: and (6) $R=4+r-p+43=46-r$ by (4); also (1) $Q=515+r$, and (3) $q=Q+p+1=517+3r$;

$\therefore$ (2) $(2r+1)+(517+3r)+r=776$, whence $r=43$, &c.

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**19.** 3. Observing that

$$12x-8=2\,(6x-4)=2\,[\sqrt{\{2(x+2)\}}+2\sqrt{(2-x)}]\,[\sqrt{\{2(x+2)\}}-2\sqrt{(2-x)}],$$

we get $\sqrt{\{2\,(x+2)\}}-2\sqrt{(2-x)}=0$, whence $x=\frac{2}{3}$,

and also $\sqrt{(9x^2+16)}=2\,\{\sqrt{2}\,(x+2)+2\sqrt{(2-x)}\}$,

by squaring which we get $9x^2+8x-32=16\sqrt{\{2\,(4-x^2)\}}$, and, squaring again,

$$81x^4+144x^3=512x-1024, \text{ or } (9x^2+8x)^2=64\,(x+4)^2,$$

whence $9x^2+8x=\pm\,8\,(x+4)$, &c.

4. From (1) $y^2=a-\frac{a}{b}\,(x^2-x)=a-\frac{a}{b}\,z$, suppose: and (2) gives

$$(x^4-2x^3+x^2)-(x^2-x)+y^2\,(x^2-x)+y^4=z^2-z+y^2z+y^4$$

$$=z^2-z+az-\frac{a}{b}\,z^2+a^2-\frac{2a^2}{b}\,z+\frac{a^2}{b^2}\,z^2=a^2:$$

$$\therefore\ (a^2-ab+b^2)\,z^2=(2a^2b-ab^2+b^2)\,z,$$

whence $z=0$ or $\dfrac{\{2a^2-(a-1)\,b\}\,b}{a^2-ab+b^2}=p$ suppose.

If $z=x^2-x=0$, then $x=0$ or $1$, $y^2=a$: if $z=p$, then

$x^2-x-p=0$, and $x=\frac{1}{2}\,\{1\pm\sqrt{(1+4p)}\}$, $y=a-\frac{a}{b}\,p$.

5. Let $x$ = rate in yards of first boat A per minute, $y$ of B; then $x-\frac{440}{3}$ = gain in yards per minute of A on spectator, and $y-x$ = ditto of B on A: $\therefore$ time in which A overtakes spectator $=\dfrac{220}{x-\frac{440}{3}}$, in which time B gains 10 yards on A;

$$\therefore\ \frac{220}{x-\frac{440}{3}}\,(y-x)=10: \quad (1)$$

and $\dfrac{2200}{x}$ = time in which A goes 1¼ mile, in which time B gains on it 30 yards;

$$\therefore\ \frac{2200}{x}\,(y-x)=30, \quad (2)$$

from which equations $x=\dfrac{4400}{21}$, and $\dfrac{2200}{x}=10\frac{1}{2}$.

6. Let $x$ = n° of comp. candles in a lb: then, since each wax candle weighs as much as $1\frac{1}{4}=\frac{5}{4}$ comp. candles, $\therefore\ \frac{4}{5}x$ = n°

of wax candles in a lb; and if $y$ = time each comp. candle burns, then $\frac{3}{2}y$ = ditto for wax. Now a lb. of comp. burns for time $xy$, and a lb. of wax for time $\frac{6}{5}xy$;

$\therefore \frac{6}{5}xy : xy :: 30d$ : price of lb. of comp. $= \frac{5}{6} \times 30d = 25d$.

7. Let $x$ = n° of gals thrown by A, $y$ = ditto by B, $z$ = ditto passing by the leak, per hour: then (since contents of cistern = 512 cub. ft.) A and B work together for time $\frac{256}{x+y-z}$, then A for time $\frac{128}{x-z}$, then the leak alone runs for time $\frac{128}{z}$, then B works for time $\frac{128}{y-z}$, again the leak runs for time $\frac{\frac{3}{2} \times \frac{1}{12} \times 64}{z} = \frac{8}{z}$, and, lastly, A and B (*without the leak*) work for time $\frac{136}{x+y}$ = (by question) $\frac{128}{x+y-z}$. (1)

$$\text{Also } \frac{256}{x+y-z} + \frac{128}{x-z} + \frac{128}{z} + \frac{128}{y-z} + \frac{8}{z} + \frac{128}{x+y-z} = \frac{31}{3}; \quad (2)$$

$$\text{and } \frac{128}{y-z} = \frac{256}{x-z}, \ (3) \quad \text{whence } x + z = 2y:$$

but (1) $x + y = 17z$; $\therefore x = 11z$, $y = 6z$, and (2) gives $z = \frac{96}{5}$.

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**20.** 2. From (1) $y + z = \sqrt{\frac{1+a^3}{x}}$, from (2) $y - z = \frac{3}{2} - x$;

$$\therefore y = \frac{1}{2}\{\sqrt{\frac{1+a^3}{x}} + (\frac{3}{2} - x)\}, \quad z = \frac{1}{2}\{\sqrt{\frac{1+a^3}{x}} - (\frac{3}{2} - x)\},$$

$$yz = \frac{1}{4}\{\frac{1+a^3}{x} - (\frac{3}{2} - x)^2\} = \frac{3}{16}, \quad \text{whence } (x-1)^3 = a^3.$$

3. Put $(x^3 - 16)^{\frac{1}{4}} = z$; then $4z^3 + 32 = z^4 + 16 + 16z$,

$$\text{or } z^4 - 16 = 4z(z^2 - 4), \ \&c.$$

4. From (1) $x^6y - y^2x^3 = x^3y(x^3 - y) = x^3 - y$:

$$\therefore x^3 - y = 0 \text{ or } x^3y = 1, \text{ that is, } y = x^3 \text{ or } x^{-3}:$$

$$\text{from (2) } y^3 + y^{-3} = 9(x + x^{-1}), \text{ or } x^9 + x^{-9} = 9(x + x^{-1});$$

$$\therefore x^9 + x^{-9} + 3(x^3 + x^{-3}) = 3\{(x^3 + x^{-3}) + 3(x + x^{-1})\},$$

$$\text{whence } x^3 + x^{-3} = \sqrt[3]{3}(x + x^{-1});$$

$$\text{and } \therefore x + x^{-1} = 0, \text{ whence } x = \pm\sqrt{-1}, \text{ or } x^2 - 1 + x^{-2} = \sqrt[3]{3},$$

$$\text{whence } x + x^{-1} = \sqrt{(\sqrt[3]{3} + 3)}, \quad x - x^{-1} = \sqrt{(\sqrt[3]{3} - 1)}.$$

5. Let $x$ = amount of money, $y$ of debts: then $\frac{1}{96}x$ = n° of cents purchased, and $\frac{92}{96}x = \frac{23}{24}x$ = produce of sale;

$\therefore$ heir's loss $= \frac{1}{24}x + \frac{1}{6}y$ = (by question) $\frac{1}{6}\{\frac{23}{24}x + \frac{5}{6}y\}$, (1)

and $\frac{1}{6}y = 140$. (2)

6. Let $x =$ n° of intruders; $5x =$ whole number in boat, and $5x - 1 =$ n° of *passengers;* $\therefore$ if $y =$ n° of minutes the last man was immersed, we have

$$5x = \tfrac{5}{6}(5x - 1) + \tfrac{1}{3}y; \quad (1)$$

and if $r =$ n° immersed at the end of $3'$, then

$$\frac{y-3}{1} + \frac{y-3+r}{r} + \frac{y-3+2r}{r^2}$$

$=$ n° extricated in the first $3' = 5x - r$; (2)

$$\text{also } 5(x-4) - 1 = 2r. \quad (3)$$

From (3) $5x = 2r + 21$, from (1) $y = \tfrac{5}{2}(x+1) = r + 13$; whence (2) gives $9r^2 - 13r - 10 = 0$, or $r = 2$.

7. Let $x, y, z$ be the n° of notches made by A, B, C: $p =$ n° of byes in A and B's, $q$ in B and C's, $n =$ n° of notches the second party were behind; then $p =$ B's notches in C's innings; $\therefore y - p =$ ditto in A's $=$ (by question) $p + z + q$, (1) and $x + (y-p) + p$ or $x + y =$ whole n° gained while A and B are in together $= \tfrac{5}{8}n$; (2) also $p + (y-p) : q + z :: q : p$, (3) $z - 1 = p$, (4) $p + z + q = \tfrac{3}{8}n - 3$. (5) But, on 2nd supposition, B's notches in A's innings $= q$, and in C's $= x + q + p$; $\therefore x + q + x + q + p + z = n - 3$. (6) From (4) $p = z - 1$; $\therefore$ from (1) $q = y - 3z + 2$, and (5) $\tfrac{3}{8}n = y - z + 4$;

$$\therefore \text{from (2) } x + y = \tfrac{5}{8}n = \tfrac{5}{3}(y - z + 4), \quad (a)$$

$$\text{and (6) } 2x + 2y - 4z + 3 = n - 3 = \tfrac{8}{3}(y - z + 4) - 3, \quad (\beta)$$

from which equations, $(a)$ and $(\beta)$, we get $x = 3z - 2$, $y = 7z - 13$; so that we have now $p = z - 1$, $q = 4z - 11$, $n = 8(2z - 3)$, and then (3) or $py = q(q + z)$ gives

$$(z-1)(7z-13) = (4z-11)(5z-11), \text{ whence } z = 4.$$

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**21.** 1. The second fraction $= \dfrac{9(9x^2-1)}{(3x-1)(x+3)} = \dfrac{9(3x+1)}{x+3}$, the factor struck out giving $3x - 1 = 0$, or $x = \tfrac{1}{3}$: the rest is easy.

2. From (1) and (2) $xy$ and $z$ are roots of $m^2 - 5m + 4 = 0$; $\therefore xy = 1, z = 4$, or $xy = 4$, $z = 1$: put successively

$$y = \frac{1}{x}, \quad y = \frac{4}{x} \text{ in (3).}$$

3. Simplifying, $2x^3 + 5x^2 - 4x - 3 = 0$, whence

$$x(2x^2 + 5x + 2) = 6x + 3, \text{ or } x(2x+1)(x+2) = 3(2x+1).$$

4. From (2)

$$(2y^2-x^2)^2-4xy(2y^2-x^2)+4x^2y^2=0, \text{ or } 2y^2-x^2=2xy:$$

$\therefore\ x^2+2xy+y^2=3y^2$, or $x+y=y\sqrt{3}$, whence $x=y(\sqrt{3}-1)$, &c.

5. Let $x=$ n° of acres; then $3x=$ rent-charge in shillings, and $\frac{3}{10}x-120=$ rates; $\therefore\ \frac{3}{5}x-240=\frac{9}{20}x.$

6. Let $x=$ n° of passengers at first, $y=$ fare of each; then $\frac{11}{12}(x+4)y=$ whole profit in pence $=55$, and

$$(x+4)y=60 \quad (1) = \text{whole fare}:$$

$$\text{and } xy+4\{\tfrac{1}{2}(x+4)\}=2(xy-4y)+32. \quad (2)$$

7. $x=$ A's rate, $y=$ B's, $\therefore 2y=$ horse's; then, since A walks *four* and rides *three* miles while B rides *four* and walks *three* miles, $\dfrac{4}{x}+\dfrac{3}{2y}=\dfrac{4}{2y}+\dfrac{3}{y}=\dfrac{5}{y}$, or $7x=8y$: (1)

also if $2z+1$ represent the n° of miles after the 7th, (since the whole number of miles is *odd*,) then

$$\frac{z+1}{x+\frac{1}{2}}+\frac{z}{2x+1}=\frac{z+1}{2x+1}+\frac{z}{y+\frac{1}{2}}, \text{ or } \frac{2z+1}{2x+1}=\frac{2z}{2y+1}; \quad (2)$$

and time of first 7 miles (taking B's expression for it in (1)) was $\dfrac{5}{y}$, time of last $2z+1$ miles (taking A's expression for it in (2)) was $\dfrac{3z+2}{2x+1}$; $\therefore\ \dfrac{5}{y}+\dfrac{3z+2}{2x+1}=2\frac{62}{63}$. (3)

From (1) $x=\frac{8}{7}y$; from (2) $\dfrac{2z+1}{2z}=\dfrac{2x+1}{2y+1}=\dfrac{16y+7}{14y+7}$,

$$\text{whence } z=\frac{14y+7}{4y}, \text{ and } 3z+2=\frac{50y+21}{4y};$$

whence (3) gives $y=3\frac{1}{2}$.

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**22.** 1. $$\frac{1}{(x^2-x-8)+12x}+\frac{1}{x^2+2x-8}+\frac{1}{(x^2-x-8)-12x}$$

$$=\frac{2(x^2-x-8)}{(x^2-x-8)^2-144x^2}+\frac{1}{x^2+2x-8}=\frac{2z}{z^2-144x^2}+\frac{1}{z+3x}=0$$

where $z=x^2-x-8$; hence $z=6x$, or $-8x$, &c.

2. From (1), $x^2+y^2+z^2-2(2x+3y+z)+4+9+1=24$,

or by (3), $x^2+y^2+z^2=70$: $(\alpha)$

$\therefore\ (\alpha)+2\times(3)$ gives $(x+y+z)^2=70+2\times 63=196$,

and $x+y+z=\pm 14$: $(\beta)$

hence $(3)-(\beta)$ gives $x+2y=16$ or $44$, and $y=8-\frac{1}{2}x$ or $22-\frac{1}{2}x$;

$3\times(\beta)-(3)$ gives $x+2z=12$ or $-72$, and $z=6-\frac{1}{2}x$ or $-36-\frac{1}{2}x$:

$\therefore$ by $(\alpha)$, $x^2+(8-\frac{1}{2}x)^2+(6-\frac{1}{2}x)^2=70$, &c.

3. Adding and subtracting numerator and denominator, we get

$$\frac{n(a^4+x^4)-a^2x^2}{a^4+x^4-na^2x^2}=\frac{n(a^4+x^4)-a^2x^2}{n(a^4+x^4)-(4n-1)a^2x^2}:$$

∴ either $n(a^4+x^4)-a^2x^2=0$, and, consequently,

$$\frac{x^2}{a^2}=\frac{1\pm\sqrt{(1-4n^2)}}{2n}=\frac{2\pm2\sqrt{(1-4n^2)}}{4n}, \text{ whence } \pm\frac{x}{a}=\frac{\sqrt{(1+2n)}\pm\sqrt{(1-2n)}}{2\sqrt{n}};$$

or $a^4+x^4-na^2x^2=n(a^4+x^4)-(4n-1)a^2x^2$,

whence $\dfrac{x^2}{a^2}=\dfrac{3n-1\pm\sqrt{\{(5n-3)(n+1)\}}}{2(n-1)}$, &c.

4. From (1)

$$(x^4+a^2y^2)(y^4+a^2x^2)+4b^2x^3y^3+2bxy\{x(y^4+a^2x^2)+y(x^4+a^2y^2)\}$$

$$=(\text{putting } xy=z)\ z^4+a^2(x^6+y^6)+a^4z^2+4b^2z^3+2bz\{z(y^3+x^3)+a^2(x^3+y^3)\}$$

$$=(\text{by } 2)\ z^4+a^2(4c^2z^2-2z^3)+a^4z^2+4b^2z^3+2bz(2cz^2+2a^2cz)$$

$$=4(a^2-b^2)(b+c)^2z^2:$$

whence, striking out the factor $z^2$,

$$z^2-2\{a^2-2b(b+c)\}z+\{a^2-2b(b+c)\}^2=0,$$

or $z=xy=a^2-2bc-2b^2=p$, suppose:

then (2) gives $x^3+\dfrac{p^3}{x^3}=2cp$.

5. Let $x$ = value of *notes* spent between P. and T., and ∴ $540-x$ = value of those remaining, which were worth $\frac{6}{5}(540-x)$ *roubles*; ∴ $\frac{6}{5}(540-x)-432=216-\frac{6}{5}x$ = *roubles* spent between T. and M.: also $432-237=195$ = *roubles* with which he started to return $=x+(216-\frac{6}{5}x)$.

6. Let $x$ = n° of oz. of gold, $y$ of silver, $20300-(x+y)$ of copper: then n° of oz. in first alloy $=\frac{12}{11}x$; ∴ n° in second $=20300-(\frac{12}{11}x+288)=20012-\frac{12}{11}x$.
Now £3 17*s* 10½*d* = £$\frac{623}{160}$, and 5*s* 6*d* = £$\frac{11}{40}$: ∴ value of the coinage in £'s $=\frac{623}{160}\times\frac{12}{11}x+\frac{11}{40}(20012-\frac{12}{11}x)=5546\frac{29}{40}$, whence $x=11$, and n° of oz. of silver $=\frac{37}{40}(20012-\frac{12}{11}x)=18500$.

7. Let $x$ = n° of fortresses; then $\dfrac{6048}{x}$ = n° in each detachment; and $2\frac{1}{2}\times\dfrac{6048}{x}=\dfrac{15120}{x}$ = n° who died; therefore $6048-\dfrac{15120}{x}-84=\dfrac{5964x-15120}{x}$ = n° now distributed, and $\dfrac{5964x-15120}{x^2}$ = n° in each detachment; ∴ n° of killed and prisoners $=(x-4)\dfrac{5964x-15120}{x^2}-210=4186-\dfrac{15120}{x}$, whence $x=12$.

**23.** 3. Observe that

$3x^2 + \frac{4}{9} = [2(x - \frac{1}{3}) + \sqrt{\{x(x - \frac{8}{3})\}}] \times [2(x - \frac{1}{3}) - \sqrt{\{x(x - \frac{8}{3})\}}]$:
so that, by question, $(x - \frac{1}{3})^2 - \frac{25}{9} = 2(x - \frac{1}{3}) - \sqrt{\{x(x - \frac{8}{3})\}}$;

$$\text{which gives } (x^2 - \tfrac{8}{3}x) + \sqrt{(x^2 - \tfrac{8}{3}x)} - 2 = 0,$$

$$\text{whence } \sqrt{(x^2 - \tfrac{8}{3}x)} = 1 \text{ or } -2, \text{ \&c.}$$

4. From (2) $2x^2y^2 = x^2 + y^2 + 1$;
from (1) $4x^4 - 2x^2 - y^2 - 3 - x^2y^2 + 2y^4 - 2x^2y^4 = 0$
$= 4x^4 - 2x^2 - y^2 - 3 + x^2y^2 + 2y^4 - y^2(x^2 + y^2 + 1)$
$= 4x^4 - 2(x^2 + y^2 + 1) - 1 + y^4 = 4x^4 - 4x^2y^2 + y^4 - 1$;

$$\therefore 2x^2 - y^2 = \pm 1, \text{ \&c.}$$

5. Let $x$ = n° of miles from L. to R., $y$ = rate of L. B. train; then calculated time in hours

$$= \frac{x + 48}{y} + \frac{1}{2} = \text{(by question) 6 hours:} \quad (1)$$

in second case $\frac{5}{4}x$ = n° of miles on the L. B. line, and $(x + 48 + 13) - \frac{5}{4}x$ or $61 - \frac{1}{4}x$ = n° of miles on the M. C. line;

$$\therefore \frac{5x}{4y} + \frac{61 - \frac{1}{4}x}{y + 1} = 5\tfrac{39}{40}. \quad (2)$$

6. Let $x$, $y$, be sides in inches of $A$ and $B$, $r$ and $s$ the diameters of balls in $A$ and $B$;

$$\therefore x^3 = y^3 + 1216 \quad (1) \quad \text{and} \quad \frac{x}{r} = \frac{y}{s} - 12. \quad (2)$$

Now the n° of balls in the faces of $A = \left(\frac{x}{r}\right)^3 - \left(\frac{x}{r} - 2\right)^3$ or $6\frac{x^2}{r^2} - 12\frac{x}{r} + 8$, which we obtain by taking from the whole number of balls the inner cube, so as to leave only the shell, one ball thick; and the number of balls in the eight edges of two opposite faces of $B = 8(\frac{y}{s} - 1)$, and the remaining balls in the four other edges $= 4(\frac{y}{s} - 2)$;

$$\therefore \text{total number in edges} = 12\frac{y}{s} - 16;$$

$$\therefore 6\frac{x^2}{r^2} - 12\frac{x}{r} + 8 : 12\frac{y}{s} - 16 :: 7 : 22. \quad (3)$$

The whole number of balls in $A$ being $\frac{x^3}{r^3}$, they may be formed into a *hollow* square, whose side will contain

$\frac{x^3}{4r^3}+1$, and be $\therefore$ $(\frac{x^3}{4r^3}+1)r$ inches long; hence its area will be $(\frac{x^3}{4r^2}+r)^2$; and the side of the *solid* square will contain $\sqrt{\frac{x^3}{r^3}}$ balls, or $r\sqrt{\frac{x^3}{r^3}}$ inches; $\therefore$ its area will be $\frac{x^3}{r}$;

$$\text{hence } (\frac{y^3}{4s^2}+s)^2-\frac{y^3}{s} : (\frac{x^3}{4r^2}+r)^2-\frac{x^3}{r} :: 129\tfrac{181}{900} : 1. \quad (4)$$

From (3) and (2) we get $\frac{x}{r}=4$, $\frac{y}{s}=16$: in (4) put $x=4r$, $y=16s$; then we get $r=6s$; and $\therefore$ $\frac{y}{x}=\frac{16s}{4r}=\frac{2}{3}$;

$$\text{whence (1)} \quad x=12, \quad r=3, \quad s=\tfrac{1}{2}.$$

7. Let $x$ = n° of quarters, $y$ = ratio of gain to whole charge in first year; then net income of first year in £'s

$$=3x+330=249+(249+20y), \text{ or } 3x-20y=168: \quad (1)$$

on this sum he pays, as income-tax in second year,

$$\tfrac{8}{240}(3x+330)=\tfrac{1}{10}(x+110);$$

$\therefore$ cost of profit in second year $=\frac{1}{10}(x+110)+\frac{1}{2}=\frac{1}{10}x+11\frac{1}{2}$, and cost of 10 pupils $=x+115$; $\therefore$ since corn-rent in second year $=\frac{11}{4}x$, and the gross charge for pupils = (as in former year) $\frac{330}{y}$, we have net income of second year

$$=\tfrac{11}{4}x+\{\frac{330}{y}-(x+115)\}-\tfrac{1}{10}(x+110)=\tfrac{33}{20}x+\frac{330}{y}-126;$$

$\therefore$ diff. of net incomes $=\frac{330}{y}-\frac{27}{20}x-456$; $\therefore$ savings in second year $=(249+20y)+(\frac{330}{y}-\frac{27}{20}x-456)$, family exp. in second year $=18(\frac{330}{y}-\frac{27}{20}x-456)$, and repairs $=15$, the sum of which, $264+20y+19(\frac{330}{y}-\frac{27}{20}x-456)$ = net income $=\frac{33}{20}x+\frac{330}{y}-126$. (2). By substitution for $x$ from (1) in (2), we get $45y^2+2723y-1650=0$, whence $y=\frac{3}{5}$, and $\therefore$ $x=60$.

Also charge for each pupil in either year $=\frac{33}{y}=55$; and cost of living in first year $=\frac{2}{5}$ of $55=22$, and in second year $=\frac{1}{10}x+11\frac{1}{2}=17\frac{1}{2}$; $\therefore$ ratio required $=\frac{44}{35}$.

**24.** 1. $\sqrt{(x^2-3x+2)}=\sqrt{2}-\sqrt{(x^2-7x+12)}$: square &c, $x=2$ or 3.

2. $(x+\sqrt{x})^2-102(x+\sqrt{x})+51^2=25$; $\therefore\ x+\sqrt{x}=56$ or 46.

3. Put $z=\sqrt[3]{\dfrac{2x+3}{2x-3}}$; $\therefore\ \dfrac{2x}{3}=\dfrac{z^3+1}{z^3-1},\ \dfrac{4x^2+9}{4x^2-9}=\dfrac{z^6+1}{2z^3}$;

$\therefore\ z+\dfrac{1}{z}=\dfrac{z^2+1}{z}=\dfrac{8}{13}\cdot\dfrac{z^6+1}{2z^3}$; whence $z^2+1=0$ (i), $z=0$ (ii),

$13z^2=4(z^4-z^2+1)$, or $4z^4-17z^2+4=0$, whence $z=\pm 2$ or $\pm\frac{1}{2}$; (iii)

and $x=\dfrac{3}{2}\cdot\dfrac{z^3+1}{z^3-1}=\pm\frac{3}{2}\sqrt{(-1)}$ (i), or $-\frac{3}{2}$ (ii), or $\pm 1\frac{1}{6}$, $\pm 1\frac{13}{14}$ (iii).

4. Put $y=xz$; then (1) gives $a^2=x^2(1+3z)$,

$$(2)\text{ gives }\ a=x\left\{1+\frac{3(1+z)}{\sqrt{z}-1}\right\}\mp x\left\{\frac{3z+\sqrt{z}+2}{\sqrt{z}-1}\right\};$$

$\therefore\ (1+3z)(\sqrt{z}-1)^2=(3z+\sqrt{z}+2)^2$, whence $2z^2+4z^{\frac{3}{2}}+3z+2\sqrt{z}+1=0$,

or $2(z^2+2z^{\frac{3}{2}}+z)+z+2\sqrt{z}+1=(2z+1)(z+2\sqrt{z}+1)=0$;

$\therefore\ \sqrt{z}=-1,\ z=1$, or $z=-\frac{1}{2}$; and $a^2=x^2(1+3z)$, &c.

5. Let $x$ = profits required: then B's receipts were

$2\frac{1}{2}\times\frac{7}{16}x+1\times\frac{1}{2}x+1\frac{1}{2}\times\frac{1}{2}\cdot\frac{6}{5}\cdot\frac{233}{240}x=\frac{1974}{800}x=987$; $\therefore\ x=400$.

6. Let $x$ = side of cube in inches: then rain that falls in inches $=3+2\frac{29}{30}+$ &c. to 41 terms $=\frac{287}{3}$ in.; $\therefore$ whole water evaporated $=x^3+\frac{287}{3}x^2$: and water evaporated (in inches) on successive days $=\frac{30}{15}+\frac{31}{15}+$ &c. to 41 terms $=\frac{410}{3}$;

$$\therefore\ \tfrac{410}{3}x^2=x^3+\tfrac{287}{3}x^2,\ \text{or}\ x=\tfrac{123}{3}=41.$$

7. Let $x$ = A's pace at first, $y$ = height of H.; then $5+y$ = dist. from K. to top of H., which B reaches at the end of 2 hrs, when A will have gone $2x$ miles, and will be $5+y-2x$ miles behind B. Hence $2(5+y-2x)$ = space gone by A during B's first rest, and $\dfrac{2(5+y-2x)}{x}$ = time of first rest.

When B starts again, A will have gone $5+y+(5+y-2x)$ $=2(5+y-2x)+2x$ miles from K., and by the end of 3rd hour will have gone $3x$ miles; $\therefore$ space gone by A in the interval $=3x-\{2(5+y-2x)+2x\}=x-2(5+y-2x)$, and space gone by B in same interval

$$=3x-(5+y)=x-(5+y-2x);$$

$\therefore$ B's increased pace $= x \cdot \dfrac{x-(5+y-2x)}{x-2(5+y-2x)} =$ space by B in 4th hour; when A will have gone $x$ miles, and will therefore be behind B

$$x\left\{\frac{x-(5+y-2x)}{x-2(5+y-2x)}-1\right\}=\frac{x(5+y-2x)}{x-2(5+y-2x)}=\frac{2}{3}(5-x).\quad(1)$$

Since A goes $\frac{2}{3}(5-x)$ during B's second rest, the time $=\dfrac{2}{3}\cdot\dfrac{5-x}{x}$: let $z =$ dist. left to Amb.: then, since B's pace is $x \cdot \dfrac{x-(5+y-2x)}{x-2(5+y-2x)}$, the time of his reaching Amb.

$$=\frac{z}{x}\cdot\frac{x-2(5+y-2x)}{x-(5+y-2x)}=\frac{20}{7}\times\frac{2(5+y-2x)}{x}.\quad(2)$$

During this time, A will have gone $\frac{20}{7}\times 2(5+y-2x)$ miles, and in 10′ more, at the pace $\frac{6}{7}x$, he will have gone $\frac{1}{6}\times\frac{6}{7}x$, that is, altogether, $\frac{40}{7}(5+y-2x)+\frac{5}{42}x=z.\quad(3)$

Put $m=5+y-2x$;

$$\text{then (1)}\quad \frac{xm}{x-2m}=\frac{2}{3}(5-x),\quad\text{or}\quad\frac{m}{x}=\frac{10-2x}{20-x}:\quad(a)$$

$$\text{also}\quad(2)\quad\frac{7z}{5x}=\frac{8m(x-m)}{x(x-2m)}=\frac{8m}{x}+\frac{1}{6},\quad(3)$$

$$\text{whence}\quad\frac{8m}{x}\left(\frac{x-m}{x-2m}-1\right)=\frac{1}{6}=\frac{8m^2}{x(x-2m)}=\frac{8(10-2x)^2}{(20-x)3x}$$

from ($a$) by [27], and $x=4$, $y=3\frac{1}{2}$, $z=3\frac{1}{3}$, and whole distance = space by A in 4 hours $+\frac{2}{3}(5-x)+z=20$ miles.

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**25.** 2. Squaring, we shall get $x^2+3b^2-2a(a+b)=2(a+b)\sqrt{(a^2-x^2)}$; squaring again, and arranging,

$$x^4+2x^2(5b^2+2ab)=12ab^2(a+b)-9b^4;$$

$$\therefore \{x^2+(5b^2+2ab)\}^2=16b^4+32ab^3+16a^2b^2=16b^2(a+b)^2;\ \&\text{c.}$$

3. Put $y=mx$: then (i) $x=31-4m^2$, (ii) $m^3x=31m^2-4$;

$$\therefore \frac{\text{(ii)}}{\text{(i)}}=m^3=\frac{31m^2-4}{31-4m^2},\quad\text{or}\quad 31m^2(m-1)=4(m^5-1):$$

hence $m=1$, which gives $x=27=y$:

or $31m^2=4(m^4+m^3+m^2+m+1)$, or $m^4+1+(m^3+m)-\frac{27}{4}m^2=0$, a common recurring equation.

4. Put $x + y = z^3$, $x - y = v^3$, $a = c^3$; then $z + v = c$, and

$$x^2+y^2=\tfrac{1}{2}(z^6+v^6)=\tfrac{1}{2}(z^2+v^2)(z^4-z^2v^2+v^4)=\tfrac{1}{2}(c^2-2zv)\{(z^2+v^2)^2-3z^2v^2\}$$
$$=\tfrac{1}{2}(c^2-2zv)(c^4-4c^2zv+z^2v^2),\ \text{and } x^2-y^2=z^3v^3;$$

hence (2) gives $\tfrac{1}{2}(c^2-2zv)(c^4-4c^2zv+z^2v^2)=(c^2-zv)^3$,

whence, at once, by reduction, $3v^2z^2 = c^4$, and $vz = \dfrac{c^2}{\sqrt{3}}$;

$$\therefore 2x=z^3+v^3=(z+v)(z^2-zv+v^2)=(z+v)\{(z+v)^2-3zv\}=c(c^2-c^2\sqrt{3});$$

$$\therefore x=\tfrac{1}{2}a(1-\sqrt{3}),\ \text{and } 2y=z^3-v^3=(z-v)\{(z+v)^2-zv\}=(z-v)(c^2-\frac{c^2}{\sqrt{3}});$$

to find $z - v$, we have $z + v = c$, $zv = \dfrac{c^2}{\sqrt{3}}$,

$$\therefore z \text{ and } v \text{ are roots of } m^2 - cm + \frac{c^2}{\sqrt{3}} = 0,$$

where $m = \dfrac{c}{2}\{1 \pm \sqrt{(1 - \dfrac{4}{\sqrt{3}})}\}$, and $\therefore z - v = c\sqrt{(1 - \dfrac{4}{\sqrt{3}})}$,

$$\text{so that } y = \frac{a}{2}(1 - \frac{1}{\sqrt{3}})\sqrt{(1 - \frac{4}{\sqrt{3}})}.$$

5. $AB = x$, $BC = y$, $r$ = rate of first train:

$\therefore \dfrac{x}{r}$ = time to $B$, and $\dfrac{x}{r} + 7' + 2'$ = (in hours) $\dfrac{x}{r} + \dfrac{3}{20}$ = time when the trains meet: and $\dfrac{x-28}{r}$ = time the first train (A) had been in motion before the express (E) started: $\therefore \dfrac{x}{r} + \dfrac{3}{20} - \dfrac{x-28}{r} = \dfrac{3}{20} + \dfrac{28}{r}$ = time E has been in motion, and $2r(\dfrac{3}{20} + \dfrac{28}{r}) = \tfrac{3}{10}r + 56$ = distance E has gone $= y - \tfrac{1}{30}r$, (1) since $\tfrac{1}{30}r$ is the space A has advanced towards C. in the 2′ after leaving B.

But E describes $y$ in $\tfrac{3}{2}$ hrs; $\therefore \dfrac{y}{2r} = \dfrac{3}{2}$, or $y = 3r$: (2)

also time E would take from C. to A., and back again, is

$$\frac{2(x+y)}{2r} = \frac{x+y}{r} = \text{(by question) } \frac{y+28}{r} + 7' + 3' = \frac{y+28}{r} + \frac{1}{6}, \ (3)$$

where $y + 28$ is the space A has gone altogether after E. started; hence (1) and (2) $\tfrac{3}{10}r + 56 = 3r - \tfrac{1}{30}r$, or $r = 21$;

$$\therefore y = 3r = 63, \text{ and (3) } x = 28 + \tfrac{1}{6}r = 31\tfrac{1}{2}.$$

6. Let $x$ and $y$ be the increased Exports and Imports in *cents*: $p$, $q$ the original per centages in millions: $r$ the original revenue: then (1) $y = nx$:

$$\text{and (2)}\quad (p+a)\,x + (q-c)\,y = r + m;$$

$$\text{(3)}\quad (p+a')\,x + (q-c')\,y = r + m - m';$$

$$\therefore\quad (a-a')\,x + (c'-c)\,y = m';$$

$$\therefore \text{ since } y = nx,\ x = \frac{m'}{(a-a') + n(c'-c)},\ y = \frac{m'n}{(a-a') + n(c'-c)}.$$

7. Let $x$ = n° of sections = n° of electors: $\dfrac{50000}{x}$ = n° of votes in each: $y$ = n° of sections returning electors favorable to A, $x - y$ to B: then

in each of the $y$ sections, the votes for A are $\dfrac{100000}{3x}$, for B, $\dfrac{50000}{3x}$;

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Now let $z$ = n° of electoral votes for C:

$\therefore z + (z-3) + (z-14) = 3z - 17$ = whole n° of electoral votes $= x$, (1)

and $y = (x-y) + z - 19$, or $x - 2y + z = 19$: (2)

$$\text{also } y\,\frac{50000}{3x} + (x-y)\,\frac{45000}{x} = y\,\frac{100000}{3x} + (x-y)\,\frac{5000}{x} + 6000: \quad (3)$$

from (3) $y = \frac{3}{5}x$, from (1) $z = \frac{1}{3}(17 + x)$, from (2) $x = 100$, $y = 60$, $z = 39$.

THE END.

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